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THE MADISON COLLOQUIUM

1913

I. ON INVARIANTS AND THE THEORY OF NUMBERS

BY

LEONARD EUGENE DICKSON

II. TOPICS IN THE THEORY OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES

BY

WILLIAM FOGG OSGOOD

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PREFACE

Shortly after its reorganization in 1894 as a national body the American Mathematical Society inaugurated the plan of holding, at intervals of two to four years, Colloquia or courses of lectures given by representative members in their special fields. The Seventh Colloquium of the Society was held at Madison, September 10–13, 1913.

Before the Society became a national organization, a Colloquium was held at Evanston, in 1893, at which Professor KLEIN, of Göttingen, was the sole speaker.* Then followed:

THE BUFFALO COLLOQUIUM, 1896

(*a*) Professor MAXIME BÔCHER, of Harvard University: Linear Differential Equations, and Their Applications.

This Colloquium has not been published, but several papers appeared at about the time of the Colloquium, in which the author dealt with topics treated in the lectures.†

(*b*) Professor JAMES PIERPONT, of Yale University: Galois's Theory of Equations.

Published in the *Annals of Mathematics*, series 2, volumes 1 and 2 (1900).

THE CAMBRIDGE COLLOQUIUM, 1898

(*a*) Professor WILLIAM F. OSGOOD, of Harvard University: Selected Topics in the Theory of Functions.

Published in the *Bulletin of the American Mathematical Society*, volume 5 (1898), pages 59–87.

* The first edition of this Colloquium was exhausted, and a second edition was published by the Society. The title is: The Evanston Colloquium, Lectures on Mathematics; New York, American Mathematical Society, 1911.

† Two of these papers were: "Regular points of linear differential equations of the second order," Harvard University, 1896; "Notes on some points in the theory of linear differential equations," *Annals of Mathematics*, vol. 12 (1898).

(*b*) Professor Arthur G. Webster, of Clark University: The Partial Differential Equations of Wave Propagation.

The Ithaca Colloquium, 1901

(*a*) Professor Oskar Bolza, of the University of Chicago: The Simplest Type of Problems in the Calculus of Variations.

Published in amplified form under the title: Lectures on the Calculus of Variations, Chicago, 1904.

(*b*) Professor Ernest W. Brown, of Haverford College: Modern Methods of Treating Dynamical Problems, and in Particular the Problem of Three Bodies.

Beginning with the lectures of 1903, the Colloquia have been published as monographs, and are here numbered accordingly.

I. The Boston Colloquium, 1903

(*a*) Professor Henry S. White, of Northwestern University: three lectures on Linear Systems of Curves on Algebraic Surfaces.

(*b*) Professor Frederick S. Woods, of the Massachusetts Institute of Technology: three lectures on Forms of Non-Euclidean Space.

(*c*) Professor Edward B. Van Vleck, of Wesleyan University: six lectures on Selected Topics in the Theory of Divergent Series and Continued Fractions.

Published for the Society under the title: The Boston Colloquium Lectures on Mathematics. New York, The Macmillan Company, 1905.

II. The New Haven Colloquium, 1906

(*a*) Professor Eliakim H. Moore, of the University of Chicago: five lectures on an Introduction to a Form of General Analysis.

(*b*) Professor Ernest J. Wilczynski, of the University of California: four lectures on Projective Differential Geometry.

(*c*) Professor MAX MASON, of Yale University: four lectures on Selected Topics in the Theory of Boundary Value Problems of Differential Equations.

Published under the title: The New Haven Mathematical Colloquium. Yale University Press, 1910.

III. THE PRINCETON COLLOQUIUM, 1909

(*a*) Professor GILBERT A. BLISS, of the University of Chicago: four lectures on Fundamental Existence Theorems.

(*b*) Professor EDWARD KASNER, of Columbia University: four lectures on Differential-Geometric Aspects of Dynamics.

Published under the title: The Princeton Colloquium Lectures on Mathematics. New York, American Mathematical Society, 1913.

It is contemplated that the Society will henceforth regularly publish the Colloquia, and thus the present volume appears as Volume IV in the series.

IV. THE MADISON COLLOQUIUM, 1913

(*a*) Professor LEONARD E. DICKSON, of the University of Chicago: five lectures on Invariants and the Theory of Numbers.

(*b*) Professor WILLIAM F. OSGOOD, of Harvard University: five lectures on Topics in the Theory of Functions of Several Complex Variables.

ON INVARIANTS AND THE THEORY OF NUMBERS

BY

LEONARD EUGENE DICKSON

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ON INVARIANTS AND THE THEORY OF NUMBERS

BY

LEONARD EUGENE DICKSON

INTRODUCTION

A simple theory of invariants for the modular forms and linear transformations employed in the theory of numbers should be of an importance commensurate with that of the theory of invariants in modern algebra and analytic projective geometry, and should have the advantage of introducing into the theory of numbers methods uniform with those of algebra and geometry.

In considering the invariants of a modular form (a homogeneous polynomial with integral coefficients taken modulo p, where p is a prime), we see at once that the rational integral invariants of the corresponding algebraic form with arbitrary variables as coefficients give rise to as many modular invariants of the modular form, and that there are numerous additional invariants peculiar to the case of the theory of numbers. Moreover, nearly all of the processes of the theory of algebraic invariants, whether symbolic or not, either fail for modular invariants or else become so complicated as to be useless. For instance, the annihilators are no longer linear differential operators. The attempt to construct a simple theory of modular invariants from the standpoints in vogue in the algebraic theory was a failure, although useful special results were obtained in this laborious way. Later I discovered a new standpoint which led to a remarkably simple theory of modular invariants. This standpoint is of function-theoretic character, employing the

simple successful methods. Finally, it may be remarked that the present theory is equally simple when the coefficients of the forms and linear transformations are not integers, but are elements of any finite field.

I am much indebted to Dr. Sanderson and Professors Cole and Glenn for reading the proof sheets.

values of the invariant, and using linear transformations only in the preliminary problem of separating into classes the particular forms obtained by assigning special values to the coefficients of the ground form. As to the practical value of the new theory as a working tool, it may be observed that the problem to find a fundamental system of modular seminvariants of a binary form is from the new standpoint a much simpler problem than the corresponding one in the algebraic case; indeed, we shall exhibit explicitly the fundamental system of modular seminvariants for a binary form of general degree.

It will now be clear why these Lectures make no use of the technical theories of algebraic invariants. On the contrary, they afford an introduction to that subject from a new standpoint and, in particular, throw considerable new light on the relations between the subjects of rational integral invariants and transcendental invariants of algebraic forms and the corresponding questions for seminvariants. Again, I shall make no use of technical theory of numbers, presupposing merely the concepts of congruence and primitive roots, Fermat's theorem, and (in Lectures III and V) the concept of quadratic residues.

The developments given in these Lectures are new, with exceptions in the case of Lecture I, which presents an introduction to the theory, and in the case of the earlier and final sections of Lecture III. But in these cases the exposition is considerably simpler and more elementary than that in my published papers on the same topics. The contacts with the work of other writers will be indicated at the appropriate places. Much light is thrown upon the unsolved problem of Hurwitz concerning formal invariants.

In many parts of these Lectures, I have not aimed at complete generality and exhaustiveness, but rather at an illumination of typical and suggestive topics, treated by that particular method which I have found to be the best of various possible methods. Surely in a new subject in which most of the possible methods are very complex, it is desirable to put on record an account of the

transformation

$$x_1 = \alpha_1^{-\frac{1}{2}}x_1', \quad x_m = \alpha_1^{\frac{1}{2}}x_m', \quad x_i = x_i' \quad (i = 2, \cdots, m-1)$$

of determinant unity. Hence there exists a linear transformation with complex coefficients of determinant unity which replaces q_m by

$$(4) \qquad x_1^2 + \cdots + x_{m-1}^2 + Dx_m^2, \qquad x_1^2 + \cdots + x_r^2,$$

according as $r = m$ or $r < m$. In the first case, the final coefficient is D since the determinant (2) of a form q_m equals that of the form derived from q_m by any linear transformation of determinant unity. Hence all quadratic forms (1) may be separated into the classes

$$(5) \qquad C_{m,D}, \quad C_r \quad (D \neq 0, r = 0, 1, \cdots, m-1),$$

where, for a particular number $D \neq 0$, the class $C_{m,D}$ is composed of all forms q_m of determinant D, each being transformable into (4_1); while, for $0 < r < m$, the class C_r is composed of all forms of rank r, each being transformable into (4_2); and, finally, the class C_0 is composed of the single form with every coefficient zero. In the last case, the determinant D is said to be of rank zero. Using also the fact that the rank of the determinant of a quadratic form is not altered by linear transformation, we conclude that *two quadratic forms are transformable into each other by linear transformations of determinant unity if and only if they belong to the same class* (5).

2. *Single-valued Invariants of* q_m.—Using the term function in Dirichlet's sense of correspondence, we shall say that a single-valued function ϕ of the undetermined coefficients β_{ij} of the general quadratic form q_m is an *invariant* of q_m if ϕ has the same value for all sets $\beta'_{ij}, \beta''_{ij}, \cdots$ of coefficients of forms $q'_m, q''_m, \cdots$ belonging to the same class.* The values $v_{m,D}$, v_r of ϕ for the various classes (5) are in general different. For example, the determinant D is an invariant; likewise the single-valued func-

* Briefly, if ϕ has the same value for all forms in any class.

LECTURE I

A THEORY OF INVARIANTS APPLICABLE TO ALGEBRAIC AND MODULAR FORMS

Introduction to the Algebraic Side of the Theory by Means of the Example of an Algebraic Quadratic Form in m Variables, §§ 1–3

1. *Classes of Algebraic Quadratic Forms.*—Let the coefficients of

$$q_m = \sum_{i,j=1}^{m} \beta_{ij} x_i x_j \qquad (\beta_{ji} = \beta_{ij}) \tag{1}$$

be ordinary real or complex numbers. Let the determinant

$$D = |\beta_{ij}| \qquad (i, j = 1, \cdots, m) \tag{2}$$

of a particular form q_m be of rank r $(r > 0)$; then every minor of order exceeding r is zero, while at least one minor of order r is not zero. There exists a linear transformation of determinant unity which replaces this q_m by a form*

$$\alpha_1 x_1^2 + \cdots + \alpha_r x_r^2 \quad (\alpha_1 \neq 0, \cdots, \alpha_r \neq 0). \tag{3}$$

Indeed, if $\beta_{11} \neq 0$, we obtain a form lacking $x_1x_2, \cdots, x_1x_m$ by substituting

$$x_1 - \beta_{11}^{-1}(\beta_{12}x_2 + \cdots + \beta_{1m}x_m)$$

for x_1. If $\beta_{11} = 0$, $\beta_{ii} \neq 0$, we substitute x_i for x_1 and $-x_1$ for x_i; while, if every $\beta_{kk} = 0$, and $\beta_{12} \neq 0$, we substitute $x_2 + x_1$ for x_2; in either case we obtain a form in which the coefficient of x_1^2 is not zero. We now have $\alpha_1 x_1^2 + \phi$, where $\alpha_1 \neq 0$ and ϕ involves only $x_2, \cdots, x_m$. Proceeding similarly with ϕ, we ultimately obtain a form (3).

Now (3) is replaced by a similar form having $\alpha_1 = 1$ by the

---

* Note for later use that each α_k and each coefficient of the transformation is a rational function of the β's with integral coefficients.

multiplies a particular x_i $(i < m)$ by ρ^k and x_m by ρ^{-k} is of determinant unity.

First, let $r < m$. Applying transformations of the last type to (3), we obtain

$$(6) \qquad x_1^2 + \cdots + x_s^2 + \rho x_{s+1}^2 + \cdots + \rho x_r^2.$$

Under the transformation of determinant unity

$$x_i = \alpha X_i + \beta X_j, \quad x_j = -\beta X_i + \alpha X_j, \quad x_m = (\alpha^2 + \beta^2)^{-1} X_m,$$

$x_i^2 + x_j^2$ becomes $(\alpha^2 + \beta^2)(X_i^2 + X_j^2)$. Choose* integers α, β so that

$$(7) \qquad \rho(\alpha^2 + \beta^2) \equiv 1 \pmod{p}.$$

Hence the sum of two terms of (6) with the coefficient ρ can be transformed into a sum of two squares. Thus by means of a linear transformation, with integral coefficients of determinant unity, q_m can be reduced to one of the forms

$$(8) \quad x_1^2 + \cdots + x_{r-1}^2 + x_r^2, \quad x_1^2 + \cdots + x_{r-1}^2 + \rho x_r^2 \quad (0 < r < m).$$

Next, let $r = m$. We obtain initially

$$x_1^2 + \cdots + x_s^2 + \rho x_{s+1}^2 + \cdots + \rho x_{m-1}^2 + \sigma x_m^2,$$

in which σ need not equal ρ as in (6). If there be an even number of terms with the coefficient ρ, we obtain as above a form of type (4_1). In the contrary case, we get

$$f = x_1^2 + \cdots + x_{m-2}^2 + \rho x_{m-1}^2 + \rho^{-1} D x_m^2.$$

If $D \equiv \rho^{2l+1} \pmod{p}$, f is transformed into (4_1) by

$$x_{m-1} = -\rho^l X_m, \quad x_m = \rho^{-l} X_{m-1}.$$

But if $D \equiv \rho^{2l}$, f is reduced to (4_1) by the transformation

$$x_{m-1} = \alpha X_{m-1} + \delta \rho^{2l-1} X_m, \quad x_m = -\delta X_{m-1} + \alpha \rho X_m,$$

$$\rho(\alpha^2 + \rho^{2l-2}\delta^2) \equiv 1,$$

---

* If $p = 5$, $\rho = 2$, we may take $\alpha = \beta = 2$. For any p, either there is an integer l such that $l^2 \equiv -1 \pmod{p}$ and we may take $\rho(\alpha + l\beta) \equiv 1$, $\alpha - l\beta \equiv 1$; or else $x^2 + 1$ takes $1 + (p-1)/2$ incongruent values modulo p, no one divisible by p, when x ranges over the integers $0, 1, \cdots, p-1$, so that $x^2 + 1$ takes at least one value of the form ρ^{2e-1}. In the latter event, $\alpha = \rho^{-e}$, $\beta = x\alpha$ satisfy (7).

tion r of the undetermined coefficients β_{ij} which specifies the rank of $|\beta_{ij}|$.

Each consistent set of values of D and r uniquely determines a class (5) and, by definition, each class uniquely determines a value of ϕ. Hence ϕ is a single valued function of D and r.

Every single-valued invariant of a system of forms is a single-valued function of the invariants (D and r in our example) which completely characterize the classes.

3. *Rational Integral Invariants of q_m.*—If the invariant ϕ is a rational integral function of the coefficients β_{ij}, it equals a rational integral function of D. For, if the β's have any values such that $D \neq 0$, ϕ has the same value for the form (1) as for the particular form (4_1) of the same class. Hence $\phi = P(D)$, where $P(D)$ is a polynomial in D with numerical coefficients. Since this equation holds for all sets of β's whose determinant is not zero, it is an identity.

Introduction to the Number Theory Side of the Theory of Invariants by Means of the Example of a Modular Quadratic Form, §§ 4–7

4. *Classes of Modular Quadratic Forms q_m.*—Let $x_1, \cdots, x_m$ be indeterminates in the sense of Kronecker. Let each β_{ij} be an integer taken modulo p, where p is an odd prime. Then the expression (1) is called a modular quadratic form. By § 1, there exists a linear transformation, whose coefficients are integers* taken modulo p and whose determinant is congruent to unity, which replaces q_m by a quadratic form (3) in which each α_k is an integer not divisible by p. Thus† each α_k is congruent to a power of a primitive root ρ of p. After applying a linear transformation of determinant unity which permutes $x_1^2, \cdots, x_r^2$, we may assume that $\alpha_1, \cdots, \alpha_s$ are even powers of ρ and that $\alpha_{s+1}, \cdots, \alpha_r$ are odd powers of ρ. The transformation which

* Perhaps initially of the form a/b, where a and b are integers, b not divisible by p. But there exists an integral solution q of $qb \equiv a \pmod{p}$.

† For $p = 5$, $\rho = 2$, $1 \equiv 2^4$, $2 \equiv 2^1$, $3 \equiv 2^3$, $4 \equiv 2^2 \pmod{5}$.

The invariants D and r therefore do not completely characterize the classes of modular quadratic forms, a result in contrast to that for algebraic quadratic forms. We shall give a criterion to decide whether a given form of rank r $(0 < r < m)$ is of class $C_{r,\,1}$ or of class $C_{r,\,-1}$ and later deduce an invariantive criterion.

5. *Criterion for Classes* $C_{r,\,\pm 1}$.—Such a criterion may be obtained from Kronecker's elegant theory of quadratic forms.* We shall make use of the theorem that a symmetrical determinant of rank r $(r > 0)$ has a non-vanishing principal minor M of order r, i. e., one whose diagonal elements lie in the main diagonal of the given determinant.† After an evident linear transformation of determinant unity, we may set

$$M = |\beta_{ij}| \not\equiv 0 \pmod{p} \qquad (i, j = 1, \cdots, r). \tag{10}$$

In the present problem, $r < m$. To q_m apply the transformation

$$x_i = X_i + c_i X_m \qquad (i = 1, \cdots, r),$$
$$x_i = X_i \qquad (i = r + 1, \cdots, m)$$

of determinant unity in which the c_i are integers. We get

$$\sum_{i,\,j=1}^{m-1} \beta_{ij} X_i X_j + 2 \sum_{j=1}^{m-1} B_{jm} X_j X_m + \left(\sum_{j=1}^{r} B_{jm} c_j + B_{mm} \right) X_m^{\,2},$$

where

$$B_{jm} = \sum_{i=1}^{r} \beta_{ij} c_i + \beta_{jm} \qquad (j = 1, \cdots, m).$$

In view of (10) there are unique values of $c_1, \cdots, c_r$ such that

$$B_{jm} \equiv 0 \pmod{p} \qquad (j = 1, \cdots, r).$$

But the determinant of the coefficients of $c_1, \cdots, c_r, 1$ in

$$B_{1m}, \quad B_{2m}, \quad \cdots, \quad B_{rm}, \quad B_{km} \qquad (r < k \leqq m)$$

* Kronecker, Werke, vol. 1, p. 166, p. 357; cf. Gundelfinger, *Crelle*, vol. 91 (1881), p. 221; Bôcher, Introduction to Higher Algebra, p. 58, p. 139.

† The most elementary proof is that by Dickson, *Annals of Mathematics*, ser. 2, vol. 15 (1913), pp. 27, 28. For other short proofs, see Wedderburn, ibid., p. 29, and Kowalewski, Determinantentheorie, pp. 122–124.

of determinant unity. The final condition is of the form (7) with $\beta = \rho^{l-1}\delta$ and hence has integral solutions α, δ.

Hence the classes of modular quadratic forms are

$$C_{m,\,D},\quad C_{r,\,1},\quad C_{r,\,-1},\quad C_0 \tag{9}$$
$$(D = 1, \cdots, p-1;\ r = 1, \cdots, m-1),$$

where $C_{m,\,D}$ is composed of all modular quadratic forms whose determinant is a given integer D not divisible by p, each being transformable into (4_1), where $C_{r,\,1}$ and $C_{r,\,-1}$ are composed of all forms transformable into (8_1) and (8_2) respectively, and C_0 is composed of the form all of whose coefficients are zero.

Two modular quadratic forms are transformable into each other by linear transformations with integral coefficients of determinant unity modulo p if and only if they belong to the same class (9). Indeed, since D and r are invariants,* it remains only to show that the two forms (8) are not transformable into each other.† But if a linear transformation

$$x_i = \sum_{j=1}^{m} \alpha_{ij} X_j \qquad (i = 1, \cdots, m)$$

replaces $f = x_1^2 + \cdots + x_r^2$ by $F = X_1^2 + \cdots + X_{r-1}^2 + \rho X_r^2$, then, for $j > r$,

$$\frac{\partial f}{\partial X_j} \equiv 2\sum_{i=1}^{r} x_i \frac{\partial x_i}{\partial X_j} = \frac{\partial F}{\partial X_j} = 0, \quad \frac{\partial x_i}{\partial X_j} = \alpha_{ij} = 0 \qquad (i \leqq r,\, j > r),$$

$$x_i = \sum_{j=1}^{r} \alpha_{ij} X_j \qquad (i = 1, \cdots, r).$$

Hence under this partial transformation on $x_1, \cdots, x_r$, we would have $f = F$. Thus the determinant of F would equal $|\alpha_{ij}|^2$ times the determinant unity of f and hence equal an even power of ρ. But the determinant of F is actually ρ.

* r is now the maximum order of a minor not divisible by p.

† An immediate proof follows from the values taken by the invariant A_r given below. But as the necessity of constructing A_r is based upon the fact that the forms (8) do not belong to the same class, it seems preferable to prove the last fact without the use of A_r.

$$(13)\qquad \begin{aligned} A_r = \{M_1^{\frac{p-1}{2}} + M_2^{\frac{p-1}{2}}(1 - M_1^{p-1}) + \cdots \\ + M_n^{\frac{p-1}{2}}(1 - M_1^{p-1}) \cdots (1 - M_{n-1}^{p-1})\}\Pi(1 - d^{p-1}), \end{aligned}$$

where $M_1, \cdots, M_n$ denote the principal minors of order r taken in any sequence, and d ranges over the principal minors of orders exceeding r. For, if any $d \not\equiv 0$, the rank exceeds r and $A_r \equiv 0$ by Fermat's theorem. Next, let every $d \equiv 0$, so that the rank is r or less, and the final factor in (13) is congruent to unity. Then, if every $M_i \equiv 0$, the rank is less than r and $A_r \equiv 0$. But, if $M_1 \not\equiv 0$,

$$A_r \equiv M_1^{\frac{p-1}{2}} \equiv \pm 1 \qquad (\text{mod } p),$$

by (12), the sign being the same as in $C_{r,\,\pm 1}$. If $M_1 \equiv 0$, $M_2 \not\equiv 0$,

$$A_r \equiv M_2^{\frac{p-1}{2}} \equiv \pm 1 \qquad (\text{mod } p),$$

etc. Note for later use that

$$(14)\qquad A_m = D^{\frac{p-1}{2}}.$$

7. *Rational Integral Invariants of* q_m.—The function

$$(15)\qquad I_0 = \Pi(1 - \beta_{ij}^{p-1}) \qquad (i, j = 1, \cdots, m;\ i \leqq j)$$

has the value 1 for the form (of class C_0) all of whose coefficients are zero and the value 0 for all remaining forms q_m, and hence is an invariant of q_m. We now have rational integral invariants

$$(16)\qquad D, \quad A_1, \quad \cdots, \quad A_{m-1}, \quad I_0$$

which completely characterize the classes (9). Hence, by the general theorem in § 12, any rational integral invariant of the modular form q_m is a rational integral function of the invariants (16) with integral coefficients. In other words, invariants (16) form a fundamental system of rational integral invariants of q_m.

If we employ not merely, as before, linear transformations with integral coefficients of determinant unity modulo p, but those of all determinants, we obtain at once the classes

$$C_{r,\,\pm 1}, \quad C_0 \qquad (r = 1, \cdots, m),$$

is the minor of β_{km} in the determinant

$$|\beta_{ij}| \qquad (i, j = 1, \cdots, r, k, m)$$

and hence is zero, being of order $r + 1$. Hence $B_{km} \equiv 0$. Thus q_m has been transformed into

$$\sum_{i,j=1}^{m-1} \beta_{ij} X_i X_j.$$

After repetitions of this process, q_m is transformed into*

$$\sum_{i,j=1}^{r} \beta_{ij} x_i x_j. \tag{11}$$

This form, of determinant M, can be reduced (§ 4) to

$$x_1^2 + \cdots + x_{r-1}^2 + M x_r^2$$

by a linear transformation on $x_1, \cdots, x_r$ with integral coefficients of determinant unity modulo p. Express M as a power $\rho^{2l+\epsilon}$ ($\epsilon = 0$ or 1) of a primitive root. Since $r < m$, we may replace x_r by $\rho^{-l}x_r$ and x_m by $\rho^l x_m$ and obtain (8_1) or (8_2) according as $\epsilon = 0$ or $\epsilon = 1$. Now $\rho^{(p-1)/2}$ is not congruent to unity, but its square is congruent to unity modulo p, by Fermat's theorem; hence it is $\equiv -1$. Thus, in the respective cases,

$$M^{\frac{p-1}{2}} \equiv +1 \quad \text{or} \quad -1 \qquad (\text{mod } p). \tag{12}$$

Hence if a form is of rank r and if M is any chosen r-rowed principal minor not divisible by p, the form is of class $C_{r,\,1}$ or $C_{r,\,-1}$ according as the first or second alternative (12) holds.

6. *Invariantive Criterion for Classes $C_{r,\,\pm 1}$.*—A function which has the value $+1$ for any form of class $C_{r,\,+1}$, the value -1 for any form of class $C_{r,\,-1}$, and the value zero for the remaining classes $C_{m,\,D}$, C_0, $C_{k,\,\pm 1}$ ($k \neq r$), is an invariant (§ 2). This function† is

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* This proof and the results in §§ 4–13 are due to Dickson, *Transactions of the American Mathematical Society*, vol. 10 (1909), pp. 123–133.

† Constructed synthetically in the paper last cited.

For, if

$$\phi \equiv v_{e_1, \ldots, e_s} \quad \text{when} \quad c_1 \equiv e_1, \cdots, c_s \equiv e_s \qquad (\text{mod } p),$$

then ϕ is identically congruent (as to $c_1, \cdots, c_s$) to

$$(18) \qquad \sum_{e_1, \cdots, e_s = 0}^{p-1} v_{e_1, \ldots, e_s} \prod_{i=1}^{s} \{1 - (c_i - e_i)^{p-1}\},$$

as shown by Fermat's theorem.

10. *Characteristic Modular Invariants.*—The characteristic invariant I_k of the class C_k is defined to be that modular invariant which has the value unity for systems of forms of the class C_k and the value zero for any of the remaining classes.

For example, for a single quadratic form q_m, I_0 is given by (15), while the characteristic invariants for the classes $C_{r,\,1}$ and $C_{r,\,-1}$ are

$$(19) \qquad I_{r,\,1} = \tfrac{1}{2}(A_r^2 + A_r), \quad I_{r,\,-1} = \tfrac{1}{2}(A_r^2 - A_r).$$

For any system of forms with the coefficients $c_1, \cdots, c_s$, we have

$$(20) \qquad I_k = \sum \prod_{i=1}^{s} \{1 - (c_i - c_i^{(k)})^{p-1}\},$$

where the sum extends over all sets of coefficients $c_1^{(k)}, \cdots, c_s^{(k)}$ of the various systems of forms of class C_k. In particular, in accord with (15),

$$(21) \qquad I_0 = \prod_{i=1}^{s} (1 - c_i^{p-1}).$$

11. *Number of Linearly Independent Modular Invariants.*—Since any modular invariant I takes certain values $v_0, \cdots, v_{n-1}$ for the respective classes $C_0, \cdots, C_{n-1}$, we have

$$(22) \qquad I = v_0 I_0 + v_1 I_1 + \cdots + v_{n-1} I_{n-1}.$$

Hence any modular invariant can be expressed in one and but one way as a linear homogeneous function of the characteristic invariants. Moreover, *the number of linearly independent modular invariants equals the number of classes.*

and see that these are characterized by $A_1, \cdots, A_m, I_0$. The latter therefore form a fundamental system of rational integral absolute invariants. But D is a relative invariant.

General Theory of Modular Invariants, §§ 8–14

8. *Definitions.*—Let S be any system of forms in $x_1, \cdots, x_m$ with undetermined integral coefficients taken modulo p, a prime. Let G be any group of linear transformations on $x_1, \cdots, x_m$ with integral coefficients taken modulo p. The particular systems $S', S'', \cdots$, obtained from S by assigning to the coefficients particular sets of integral values modulo p, may be separated into *classes* $C_0, C_1, \cdots, C_{n-1}$ such that two systems belong to the same class if and only if they are transformable into each other by transformations of G.

A single-valued function ϕ of the coefficients of the forms in the system S is called an *invariant* of S under G if, for $i = 0, 1, \cdots, n-1$, the function ϕ has the same value v_i for all systems of forms in the class C_i.

In case the values taken by ϕ are integers which may be reduced at will modulo p and congruent values be identified, the invariant is called *modular*. Since this reduction can be effected on each coefficient of the modular forms comprising our system S, any rational integral invariant of S is a modular invariant.

An example of a non-modular invariant is the transcendental function r defining the rank of the determinant of the modular quadratic form q_m. The values of r are evidently not to be identified when merely congruent modulo p. However, the residue of r modulo p is a modular invariant, since

$$(17) \qquad r \equiv A_1^2 + 2A_2^2 + \cdots + mA_m^2 \qquad (\text{mod } p).$$

9. *Modular Invariants are Rational and Integral.*—Any modular invariant ϕ of a system S of modular forms can be identified with a rational integral function (with integral coefficients) of the coefficients $c_1, \cdots, c_s$ appearing in the forms of the system S.

terms of the c's, $P(A, \cdots, L)$ becomes a polynomial, which, after exponents are reduced below p, will be designated by $\psi(c_1, \cdots, c_s)$. Then ϕ and ψ are identically congruent in $c_1, \cdots, c_s$, that is, corresponding coefficients are congruent modulo p. In fact, a polynomial of type ϕ is uniquely determined by its values for the p^s sets of values of $c_1, \cdots, c_s$, each chosen from $0, 1, \cdots, p-1$ (§ 9). Hence ϕ can be expressed as a polynomial in $A, \cdots, L$ with integral coefficients.*

13. *Minor Rôle of Modular Covariants.*—In contrast with the case of algebraic forms, the classes of modular forms are completely characterized by rational integral invariants. Such invariants therefore suffice to express all invariantive properties of a system of modular forms. In this respect, modular covariants play a superfluous rôle. For instance, a projective property of a system of algebraic forms is often expressed by the identical vanishing of a covariant. But if C is a modular covariant with the coefficients $c_1, \cdots, c_s$, then I_0 given by (21) is a modular invariant of C and hence of the initial system of forms. We have $C \equiv 0$ or $C \not\equiv 0 \pmod{p}$ identically, according as $I_0 \equiv 1$ or $I_0 \equiv 0$.

14. *References to Further Developments.*—This general theory of modular invariants has been applied by me to determine a complete set of linearly independent modular invariants of q linear forms on m variables,† and a fundamental system of modular invariants of a pair of binary quadratic forms and of a pair of binary forms, one quadratic and the other linear.‡

The theory has been extended to combinants and applied to a pair of binary quadratic forms.§

* This correct theorem for any finite field cannot be extended at once to any field as stated by me in *American Journal of Mathematics*, vol. 31 (1909), top of p. 338.

† *Proceedings of the London Mathematical Society*, ser. 2, vol. 7 (1909), pp. 430–444.

‡ *American Journal of Mathematics*, vol. 31 (1909), pp. 343–354; cf. pp. 103–146, where a less effective method is used.

§ Dickson, *Quarterly Journal of Mathematics*, vol. 40 (1909), pp. 349–366.

For example, using (19), we see that a complete set of linearly independent modular invariants of the quadratic form q_m modulo p ($p > 2$) is given by

$$(23)\quad I_0,\ A_r,\ A_r^2\ (r = 1, \cdots, m-1), \qquad D^k\ (k = 1, \cdots, p-1).$$

12. *Fundamental Systems of Modular Invariants.*—While, by (22), the characteristic invariants $I_0, \cdots, I_{n-1}$ form a fundamental system of modular invariants of a system S of modular forms, it is usually much easier to find another fundamental system. In fact, certain invariants are usually known in advance, e. g., the invariants of the corresponding system of algebraic forms. We shall prove the following fundamental theorem:

If the modular invariants $A, B, \cdots, L$ completely characterize the classes, they form a fundamental system of modular invariants.

For example, $I_0, \cdots, I_{n-1}$ evidently completely characterize the classes and were seen to form a fundamental system.

Let $c_1, \cdots, c_s$ be the coefficients of the forms in the system S. Let each c_i take the values $0, 1, \cdots, p-1$. For the resulting p^s sets of values of the c's, let the rational integral functions $A, B, \cdots, L$ of $c_1, \cdots, c_s$ take the distinct sets of values

$$A_i, \quad B_i, \quad \cdots, \quad L_i \quad (i = 0, \cdots, n-1).$$

Thus there are n classes of systems S and by hypothesis the ith class is uniquely defined by the values $A_i, \cdots, L_i$ of our invariants. A rational integral invariant $\phi(c_1, \cdots, c_s)$ takes the same value for all systems of forms in the ith class, so that this value may be designated by ϕ_i. Now the polynomial

$$P(A, B, \cdots, L) = \sum_{i=0}^{n-1} \phi_i\{1 - (A - A_i)^{p-1}\} \cdots \{1 - (L - L_i)^{p-1}\}$$

is congruent to ϕ_i when $A \equiv A_i, \cdots, L \equiv L_i \pmod{p}$. Hence

$$\phi(c_1, \cdots, c_s) \equiv P(A, B, \cdots, L) \qquad \pmod{p}$$

for all sets of integral values of $c_1, \cdots, c_s$. In view of Fermat's theorem, we may assume that each exponent in $\phi(c_1, \cdots, c_s)$ is less than p. If we replace $A, \cdots, L$ by their expressions in

2. *The Classes of Algebraic Quartic Forms.*—Consider a quartic form f in which a_k is the first non-vanishing coefficient. Apply transformation (1) with

$$t = \frac{-a_{k+1}}{(k+1)a_k}. \tag{2}$$

We obtain a form having zero in place of the former a_{k+1}. Dropping the accents on x', y', we obtain, for $k = 0, 1, 2, 3$, the respective forms

$$(3) \quad a_0 \neq 0: \qquad a_0x^4 + 6a_0^{-1}S_2x^2y^2 + 4a_0^{-2}S_3xy^3 + a_0^{-3}S_4y^4,$$

$$(4) \quad a_0 = 0, \quad a_1 \neq 0: \qquad 4a_1x^3y + a_1^{-1}S_{13}xy^3 + a_1^{-2}S_{14}y^4,$$

$$(5) \quad a_0 = a_1 = 0, \quad a_2 \neq 0: \qquad 6a_2x^2y^2 + \tfrac{1}{3}a_2^{-1}S_{24}y^4,$$

$$(6) \quad a_0 = a_1 = a_2 = 0, \quad a_3 \neq 0: \qquad 4a_3xy^3,$$

$$(7) \quad a_0 = a_1 = a_2 = a_3 = 0: \qquad a_4y^4,$$

no transformation having been made in the last case. Here

$$S_2 = a_0a_2 - a_1^2, \quad S_3 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3, \tag{8}$$

$$S_4 = a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4, \tag{9}$$

$$\begin{gathered} S_{13} = 4a_1a_3 - 3a_2^2, \quad S_{14} = a_1^2a_4 - 2a_1a_2a_3 + a_2^3, \\ S_{24} = 3a_2a_4 - 2a_3^2. \end{gathered} \tag{10}$$

If we apply to one of the forms (3)–(6) a transformation (1) with $t \neq 0$, we obtain a form having an additional (second) term. Hence no two of the forms (3)–(7) can be transformed into each other by a transformation (1), so that each represents a class of forms. For example, there is a class (5) for each set of values of the parameters a_2 and S_{24} ($a_2 \neq 0$).

3. *Rational Integral Seminvariants of an Algebraic Quartic.*—First, a_0 is a seminvariant since it has a definite value $\neq 0$ for any form in any class (3) and the value zero for any form in any class (4)–(7). Next, S_2, S_3, S_4 are seminvariants, since they have constant values

$$S_2 = -a_1^2, \quad S_3 = 2a_1^3, \quad S_4 = -3a_1^4 \qquad (\text{if } a_0 = 0) \tag{11}$$

LECTURE II

SEMINVARIANTS OF ALGEBRAIC AND MODULAR BINARY FORMS

INTRODUCTORY EXAMPLE OF THE BINARY QUARTIC FORM, §§ 1–6

1. *Comparative View.*—Let the forms

$$f = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4,$$

with real or complex coefficients, be separated into classes such that two forms f are transformable into one another by a transformation of type

$$x = x' + ty', \quad y = y', \tag{1}$$

if and only if they belong to the same class. Then a single-valued function $S(a_0, \cdots, a_4)$ is called a seminvariant of f if it has the same value for all of the forms in any class.

By the repeated application of this definition and without the aid of new principles, we shall obtain a fundamental system of rational integral seminvariants of f, then on the one hand the additional single-valued seminvariant needed to form with these a fundamental system of single-valued seminvarints, and on the other hand the additional rational integral modular seminvariants needed to form with them a fundamental system of modular seminvariants of f. It is such a comparative view that we desire to emphasize here. In later sections, we shall show that it is usually much simpler to treat the modular case independently and in particular without introducing all of the algebraic seminvariants, which become very numerous and most unwieldy for forms of high degree. The rational integral seminvariants S of an algebraic form are of special importance since each is the leading coefficient of one and but one covariant, which can be found from S by a process of differentiation. For example, the seminvariant a_0 is the leading coefficient of the covariant f.

so that each has the same value for any form in a class (4); finally,

(17) $$I = 3a_2^2, \quad J = -a_2^3 \qquad (\text{if } a_0 = a_1 = 0),$$

so that each has the same value for any form in a class (5)–(7).

From ϕ we eliminate S_4 by means of (12) and then the second and higher powers of S_3 by means of (15). Thus S equals N/a_0^k, where N is a rational integral function of

(18) $$a_0, \quad S_2, \quad S_3, \quad I, \quad J,$$

of degree 0 or 1 in S_3. If $k > 0$, we may evidently assume that not every term of the polynomial N in the arguments (18) has the factor a_0. Let $P(S_2, S_3, I, J)$ denote the aggregate of the terms of N not involving a_0 explicitly. We shall prove that, if $k > 0$, N/a_0^k is then not a rational integral function of $a_0, \cdots, a_4$. For, if it be, P vanishes when $a_0 = 0$. By (11) and (16), the terms independent of a_0 in J involve a_4, while those in I, S_2, S_3 do not. Hence J does not occur in P. Then, by (11) and the term $3a_2^2$ in I, we conclude that I does not occur in P. Thus P is a polynomial in S_2 and S_3 of degree 0 or 1 in S_3 and is not identically zero. By (11), it cannot vanish for $a_0 = 0$.

Under the initial assumption that $a_0 \neq 0$, we have now proved that any rational integral seminvariant S equals a polynomial in the functions (18). The resulting equality is therefore an identity.

The seminvariants (18) *form a fundamental system of rational integral seminvariants of the algebraic quartic form.**

They are connected by the relation, or syzygy, (15).

4. *Invariantive Characterization of the Classes.*—By § 3, the classes (3) are completely characterized by the seminvariants a_0, S_2, S_3, I. These with J characterize the classes (4) having $a_0 = 0$, $a_1 \neq 0$. For, by (11), S_2 and S_3 determine a_1; while, by (16), I and J determine the remaining parameters in (4).

* The above proof differs from that by Cayley in minor details and in the method of obtaining the functions (18) and the verification that they are seminvariants (the present method being based upon the classes).

for any form in any class (4)–(7), and constant values for any form of a definite class (3), for which therefore a_0 has a definite value $\neq 0$ and $a_0^{-1}S_2, \cdots$, and hence each S_i, has a definite value. Moreover, *these seminvariants* a_0, S_2, S_3, S_4 *completely characterize the classes* (3).

Consider a quartic form f in which a_0, a_1, a_2, a_3, a_4 are arbitrary, except that $a_0 \neq 0$. Any rational integral seminvariant $S(a_0, \cdots, a_4)$ has the same value for f as for the particular form (3) in the same class as f. Hence

$$S = S\left(a_0,\, 0,\, \frac{S_2}{a_0},\, \frac{S_3}{a_0^2},\, \frac{S_4}{a_0^3}\right) = \frac{\phi(a_0, S_2, S_3, S_4)}{a_0^j},$$

where ϕ is a rational integral function of its arguments. We therefore seek such functions ϕ as are divisible by a power of a_0, and hence by (11) in which the terms involving only a_1 cancel. The function of lowest degree is evidently

$$(12) \qquad S_4 + 3S_2^2 = a_0^2 I, \quad I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2.$$

The next lowest degree is 6 and the function is

$$dS_2S_4 + eS_3^2 + (3d + 4e)S_2^3.$$

The coefficient of d is $a_0^2IS_2$, that of e is

$$(13) \qquad \begin{gathered} S_3^2 + 4S_2^3 = a_0^2 D \\ (D \equiv a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2). \end{gathered}$$

Hence for $d = 1$, $e = -1$, the function is the product of a_0^2 and

$$(14) \quad IS_2 - D = a_0J, \quad J \equiv a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3.$$

We do not retain D since it is expressible in terms of the other functions. Eliminating D between (13) and (14), we get

$$(15) \qquad S_3^2 + 4S_2^3 - a_0^2IS_2 + a_0^3J \equiv 0.$$

Now I and J are seminvariants. Indeed, if $a_0 \neq 0$, they are expressible in terms of the parameters a_0, S_i in (3) and hence each has the same value for any form in a class (3); while

$$(16) \qquad I = -S_{13}, \quad J = -S_{14} \qquad (\text{if } a_0 = 0),$$

We shall make frequent use of the abbreviation

$$(19)\qquad P_i = (1 - a_0^{p-1})(1 - a_1^{p-1}) \cdots (1 - a_i^{p-1}).$$

Then P_1S_{24}, P_2a_3 and P_3a_4 are seminvariants* since each takes the same value for all forms in any class. For the classes (5), (6), (7), their values are S_{24}, a_3 and a_4, respectively. Hence *the five seminvariants* (18) *together with* P_1S_{24}, P_2a_3 *and* P_3a_4 *completely characterize the classes and therefore form a fundamental system of rational integral seminvariants of the quartic form* f *with integral coefficients taken modulo* p, $p > 3$.

Seminvariants of a Modular Binary Form of Order n, §§ 7–13

7. *Fundamental System of Modular Seminvariants Derived by Induction from* $n - 1$ *to* n.—It is necessary to distinguish the case in which the modulus p is prime to n from the case in which p divides n. Binomial coefficients for the form are not permissible in the second case and often not in the first case (for example, if $n = 4$, $p = 3$, since $\binom{n}{2}$ is then divisible by p). Denote the form by

$$(20)\qquad F_n = A_0x^n + A_1x^{n-1}y + \cdots + A_ny^n.$$

First, let p be prime to n. For $A_0 \neq 0$, we can transform F_n into a form lacking the second term and having as coefficients the quotients of

$$(21)\qquad \begin{gathered} \sigma_2 = nA_0A_2 - \tfrac{1}{2}(n-1)A_1^2, \\ \sigma_3 = n^2A_0^2A_3 - (n-2)nA_0A_1A_2 + \tfrac{1}{3}(n-1)(n-2)A_1^3, \quad \cdots \end{gathered}$$

by powers of nA_0. These may also be obtained from (8) by identifying F_n with

$$(22)\qquad f_n = a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{2}a_2x^{n-2}y^2 + \cdots.$$

* The first is one-half the discriminant of the semicovariant

$$P_1f/y^2 \equiv P_1(6a_2x^2 + 4a_3xy + a_4y^2) \pmod{p},$$

and the last two are the seminvariants of $P_2f/y^3 \equiv P_2(4a_3x + a_4y) \pmod{p}$.

The parameter a_2 ($a_2 \neq 0$) in (5) is determined by I and J, in view of (17).

We have now gone as far as is possible in the characterization of the classes by means of rational integral seminvariants S, since the parameters S_{24}, a_3, a_4 in (5)–(7) cannot be determined by such seminvariants. Indeed,* for $a_0 = a_1 = 0$, we have $S_2 = S_3 = 0$ by (11), while I and J reduce to powers of a_2 by (17).

5. *Single-valued Seminvariants.*—We may, however, construct a single-valued seminvariant which shall determine these outstanding parameters S_{24}, a_3, a_4. To this end consider the single-valued function V defined as follows by its values in the sense of Dirichlet. We take $V = 0$ if $a_0 \neq 0$ or if $a_1 \neq 0$, and $V = S_{24}$, a_3, a_4 in the respective cases (5), (6), (7). Since V has the same value for all forms in any class, it is a seminvariant. The seminvariants (18) and V completely characterize the classes (3)–(7) and hence, by § 2 of Lecture I, form a fundamental system of single-valued seminvariants of the algebraic binary quartic form.

6. *Seminvariants of a Modular Quartic Form.*—Passing to the number theory case, let the coefficients of the quartic form f be integers taken modulo p, where p is a prime exceeding 3. The denominator in (2) is then not divisible by p, so that the classes are again (3)–(7).

By the general theory in Lecture I, it is possible to characterize all of the classes by means of rational integral seminvariants, and the latter will then form a fundamental system. In particular, we do not now require the use of such a bizarre function as that used in § 5.

* A proof of this fact, not based upon the final theorem of § 3, would afford a better insight into the nature of the last steps in § 3 and explain, in particular, why we stopped with I and J and did not consider combinations of the S_i of higher than the sixth degree in the a's. To this end, let S be a seminvariant homogeneous of total degree i, in the a's, and isobaric, of constant weight w. As well known, $4i \geqq 2w$. Thus S cannot have a term $a_3{}^l$ or $a_4{}^l$ and cannot reduce, when $a_0 = a_1 = 0$, to $a_2{}^l S_{24}{}^m$ ($m > 0$), of degree $l + 2m$ and weight $2l + 6m$.

into classes under (1), multiply each form by y and add ϕ, we obtain the classes of forms F_n for this value of A_0. Hence, *if n is divisible by p a fundamental system of modular seminvariants of F_n is given by A_0 and a fundamental system for $\overline{F}_{n-1}$.*

For example, if $n = p = 2$,

$$\overline{F}_1 = (A_0 + A_1)x + A_2y$$

can be transformed into x or A_2y by (1), according as $A_0 + A_1 \equiv 1$ or 0 (mod 2). Adding $\phi = A_0(x^2 - xy)$ to xy and A_2y^2, we obtain representatives of the classes of forms F_2. Hence the 6 classes are completely characterized by the seminvariants A_0 and those (§ 7) of $\overline{F}_1$, and hence by

$$A_0, \quad A_1, \quad J \equiv (1 + A_0 + A_1)A_2. \tag{27}$$

9. *Seminvariants of the Binary Cubic Form.*—The classes of algebraic forms f_3 are

$$a_0x^3 + 3a_0^{-1}S_2xy^2 + a_0^{-2}S_3y^3, \tag{28}$$

$$3a_1x^2y + \tfrac{1}{4}a_1^{-1}S_{13}y^3, \quad 3a_2xy^2, \quad a_3y^3, \tag{29}$$

where the S's are given by (8) and (10_1). The discriminant D of f_3 is given by (13). As in § 3, a_0, S_2, S_3, D form a fundamental system of seminvariants of f_3; they are connected by the syzygy (13).

Henceforth, let the coefficients of f_3 be integers taken modulo p, the excluded case $p = 3$ being treated in § 15. If $p > 3$, the classes are again (28) and (29), and a fundamental system of seminvariants is given by

$$a_0, \quad S_2, \quad S_3, \quad D, \quad P_1a_2, \quad P_2a_3. \tag{30}$$

It is instructive to compare this result with that obtained by the method of § 7. Forming the functions (24) for

$$f_2' = P_0f_3/y \equiv 3P_0a_1x^2 + 3P_0a_2xy + P_0a_3y^2 \pmod{p},$$

and deleting the factor 3 from the first and second, we get*

$$P_0a_1, \quad \delta = P_0(4a_1a_3 - 3a_2^2) = P_0S_{13}, \quad P_1a_2, \quad P_2a_3.$$

* They characterize the classes (29) of f_3 with $a_0 = 0$ and may be so derived

For p prime to n, a fundamental system of seminvariants of F_n is given by $A_0, \sigma_2, \cdots, \sigma_n$ together with a fundamental system of the particular form of order $n-1$

$$\begin{aligned}(23)\qquad F'_{n-1} &= P_0F_n/y \\ &\equiv P_0A_1x^{n-1}+P_0A_2x^{n-2}y+\cdots+P_0A_ny^{n-1} \qquad (\textit{mod } p),\end{aligned}$$

where $P_0 = 1 - A_0^{p-1}$.

Indeed, $A_0, \sigma_2, \cdots, \sigma_n$ completely characterize the classes of forms F_n with $A_0 \neq 0$. Since $yF_{n-1}' \equiv F_n$ identically, when $A_0 = 0$, the classes of forms F_n with $A_0 = 0$ are completely characterized by the seminvariants of the fundamental system for F_{n-1}'.

For example, A_0 and P_0A_1 form a fundamental system of modular seminvariants of $A_0x + A_1y$ (since these characterize the classes represented by A_0x and A_1y). The corresponding functions for

$$F_1' = P_0A_1x + P_0A_2y$$

are P_0A_1 and

$$\{1 - (P_0A_1)^{p-1}\}P_0A_2 \equiv (1 - A_1^{p-1})P_0A_2 = P_1A_2 \qquad (\text{mod } p).$$

Hence the theorem shows that, *if* $p > 2$,

$$(24)\qquad A_0, \quad 2\sigma_2 = 4A_0A_2 - A_1^2, \quad P_0A_1, \quad P_1A_2$$

form a fundamental system of modular seminvariants of F_2. For f_2, these are

$$(24')\qquad 2a_0, \quad S_2 = a_0a_2 - a_1^2, \quad P_0a_1, \quad P_1a_2.$$

8. *Order a Multiple of the Modulus.*—Next, let $n = pq$. By Fermat's theorem, $x^p - xy^{p-1}$ and hence

$$(25)\qquad \phi = A_0(x^p - xy^{p-1})^q$$

is unaltered modulo p by any transformation (1). Hence if, for each value of the seminvariant A_0, we separate the forms

$$(26)\qquad \bar{F}_{n-1} \equiv \frac{1}{y}(F_n - \phi)$$

For $p > 3$, f_3' is obtained from f_3 by replacing a_0, a_1, a_2, a_3 by

$$4a_1P_0, \quad 2a_2P_0, \quad \tfrac{4}{3}a_3P_0, \quad a_4P_0,$$

respectively. Making this replacement in the second set of seminvariants of f_3 in § 9, we obtain P_0a_1, which may be dropped in view of (31), and the last five functions (34). Hence, *for $p > 3$, a fundamental system of modular seminvariants of f_4 is given by*

$$(34) \qquad a_0, \quad S_2, \quad S_3, \quad S_4, \quad P_0S_{13}, \quad P_0S_{14}, \quad P_1S_{24}, \quad P_2a_3, \quad P_3a_4.$$

Here the three S_{ij} are given by (10). Since the functions (34) completely characterize the classes (3)–(7), we have a new proof that they form a fundamental system.

11. *Explicit Fundamental System when $p > n$.*—Instead of employing the above step by step process, we can obtain directly a fundamental system of modular seminvariants of f_n when the modulus p exceeds the order n of the binary form (22). Consider a particular f_n in which a_k is the first non-vanishing coefficient:

$$\sum_{i=k}^{n} \binom{n}{i} a_i x^{n-i} y^i \qquad (a_k \neq 0).$$

To this we apply transformation (1) and obtain

$$\sum_{i=k}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} a_i t^j x'^{\,n-i-j} y'^{\,i+j} = \sum_{l=k}^{n} A_{kl} x'^{\,n-l} y'^{\,l},$$

where we have replaced j by $l - i$ and set

$$A_{kl} = \sum_{i=k}^{l} \binom{n}{i} \binom{n-i}{l-i} a_i t^{l-i} \equiv \binom{n}{l} \sum_{i=k}^{l} \binom{l}{i} a_i t^{l-i}.$$

Take $k < n$ and give to t the value (2). Thus

$$(35) \qquad A_{kl} = \frac{a_k \binom{n}{l} \sigma_{kl}}{\{(k+1)a_k\}^{l-k}},$$

$$\sigma_{kl} \equiv \sum_{i=k}^{l} (-1)^{l-i} \binom{l}{i} (k+1)^{i-k} a_k^{\,i-k-1} a_{k+1}^{\,l-i} a_i.$$

Hence, if $p > 3$, these four functions and a_0, S_2, S_3 form a fundamental system of modular seminvariants of f_3. We may drop $P_0 a_1$ since

$$(31) \qquad P_0 S_2^{\frac{p-3}{2}} S_3 \equiv \pm 2 P_0 a_1^p \equiv \pm 2 P_0 a_1 \qquad (\text{mod } p).$$

Hence a fundamental system of seminvariants of f_3 for $p > 3$ is

$$(32) \qquad a_0, \quad S_2, \quad S_3, \quad \delta = P_0 S_{13}, \quad P_1 a_2, \quad P_2 a_3.$$

It is easy to deduce δ from the old set (30), and D from this new set.*

Finally, let $p = 2$. By § 7, a fundamental system of seminvariants for f_3 is given by a_0, S_2, S_3 and a fundamental system for f_2'. The latter system is derived from (27) by replacing A_0, A_1, A_2 by $P_0 a_1$, $P_0 a_2$, $P_0 a_3$, and hence is

$$(1 + a_0) a_1, \quad (1 + a_0) a_2, \quad (1 + a_0)(1 + a_1 + a_2) a_3.$$

We may drop $(1 + a_0) a_1 \equiv (1 + a_0) S_2$.

10. *The Binary Quartic Form.* For $p = 2$, we have

$$\bar{F}_3 = A_1 x^3 + (A_0 + A_2) x^2 y + A_3 x y^2 + A_4 y^3,$$

whose seminvariants are obtained from those of f_3 at the end of § 9. They with A_0 give a fundamental system of seminvariants of F_4:

$$A_0, \quad A_1, \quad A_1 A_3 + A_0 + A_2, \quad (1 + A_1) A_3,$$
$$A_1 A_4 + A_1 A_3 (A_0 + A_2), \quad K = (1 + A_1)(1 + A_0 + A_2 + A_3) A_4.$$

An equivalent fundamental system is†

$$(33) \qquad A_0, \quad A_1, \quad A_2 + A_3, \quad (1 + A_1) A_2,$$
$$A_1 A_4 + A_0 A_2 + A_2 A_3, \quad K.$$

$$* \; D \equiv a_0^{p-3}(S_3^2 + 4 S_2^3) - \delta S_2 \qquad (\text{mod } p).$$

For, if $a_0 \not\equiv 0$, then $\delta \equiv 0$ and this relation follows from (13); while, if $a_0 = 0$, $D = a_1^2 S_{13} = a_1^2 \delta = - S_2 \delta$. Conversely, δ can be expressed in terms of the functions (30). The above relation gives δS_2. The product of this by S_2^{p-2} is congruent to δ if $S_2 \not\equiv 0$. Also $\delta \equiv 0$ if $a_0 \not\equiv 0$. There remains the case in which $S_2 \equiv 0$, $a_0 \equiv 0$, whence $a_1 \equiv 0$, $\delta \equiv - 3 a_2^2 \equiv - 3 (P_1 a_2)^2$.

† *Annals of Mathematics*, ser. 2, vol. 15, March, 1914. I there give also a complete set of linearly independent invariants and of linear covariants

in which the symbols are defined by (8)–(10), (19) *and*

$$S_5 = a_0^4a_5 - 5a_0^3a_1a_4 + 10a_0^2a_1^2a_3 - 10a_0a_1^3a_2 + 4a_1^5,$$

$$(38)\qquad \begin{aligned} S_{15} &= 16a_1^3a_5 - 40a_1^2a_2a_4 + 40a_1a_2^2a_3 - 15a_2^4,\\ S_{25} &= 27a_2^2a_5 - 45a_2a_3a_4 + 20a_3^3,\\ S_{35} &= 8a_3a_5 - 5a_4^2.\end{aligned}$$

12. *Another Method for the Case* $p > n$.—We may formulate the method of § 7 so that it shall be free from the induction process. The classes of forms (23) with $P_0A_1 \neq 0$, and hence the classes of forms F_n with $A_0 = 0$, $A_1 \neq 0$, are characterized by the seminvariants given by the products of P_0 by the functions σ_2', $\cdots$ obtained from $\sigma_2, \sigma_3, \cdots, \sigma_{n-1}$ by increasing the subscript of each A_i by unity and replacing n by $n-1$; indeed, $P_i^2 \equiv P_i$ (mod p). When the process of deriving (23) from (20) is applied to (23), we get

$$(39)\qquad \begin{aligned} F''_{n-2} &= [1 - (P_0A_1)^{p-1}]F'_{n-1}/y \equiv (1 - A_1^{p-1})P_0F_n/y^2\\ &= P_1F_n/y^2 \equiv P_1A_2x^{n-2} + P_1A_3x^{n-3}y\\ &\qquad + \cdots + P_1A_ny^{n-2} \pmod{p}.\end{aligned}$$

The class of forms (39) with $P_1A_2 \neq 0$, and hence the classes of forms F_n with $A_0 = A_1 = 0$, $A_2 \neq 0$, are characterized by the seminvariants given by the products of P_1 by the functions σ_2'', $\cdots$ obtained from $\sigma_2', \cdots, \sigma_{n-2}'$ by increasing the subscript of each A_i by unity and replacing n by $n-1$. Finally, we obtain $P_{n-2}A_{n-1}x + P_{n-2}A_ny$, characterized by the seminvariants $P_{n-2}A_{n-1}$ and $P_{n-1}A_n$. The earlier $P_{k-1}A_k$ may be dropped (§ 11).

For example, if $n = 3$, $p > 3$, the fundamental system of F_3 is

$$A_0,\quad \sigma_2,\quad \sigma_3,\quad P_0\sigma_2' = P_0(4A_1A_3 - A_2^2),\quad P_1A_2,\quad P_2A_3.$$

Changing the notation from F_3 to f_3, we see that σ_2' becomes $3(4a_1a_3 - 3a_2^2)$, so that the resulting seminvariants are (32). We may of course apply the method directly to f_3; in S_2 we replace a_0, a_1, a_2 by $3a_1, \frac{3}{2}a_2, a_3$ and obtain $\frac{3}{4}(4a_1a_3 - 3a_2^2)$.

In particular,

$$\sigma_{kk} = 1, \quad \sigma_{kk+1} = 0, \quad \sigma_{0l} = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} a_0^{i-1} a_1^{l-i} a_i,$$

the last being the algebraic seminvariant designated earlier by S_l. It is obtained from the expansion of $(a_0 - a_1)^l$ by replacing a single a_0 in each term by a_i. Except for a numerical factor not divisible by p, σ_{kl} (for $0 < k < l - 1$) equals the S_{kl} in (10) and in (38) below.

The classes C_k of forms f_n in which a_k is the first non-vanishing a are distinguished from each other by the value of a_n if $k = n$, and if $k < n$ by the values of the parameters a_k, σ_{kl} ($l = k + 2, \cdots, n$). Employing the notation (19), we shall verify that $P_{k-1}a_k$ and $P_{k-1}\sigma_{kl}$ are modular seminvariants of f_n. They vanish for a form C_j ($j \leqq k - 1$) since then $1 - a_j^{p-1} \equiv 0$. For C_k, they reduce to the parameters a_k and σ_{kl} of that class. For $a_0 = 0, \cdots, a_k = 0$, the first is zero and the second is the expression for σ_{kl} when $a_k = 0$, whose non-vanishing terms (given by $i = k$ and $i = k + 1$) are constant multiples of a_{k+1}^{l-k}; but a_{k+1} is constant for any class C_j ($j > k$).

It follows also that the parameter a_{k+1} in a class C_{k+1} is determined by the seminvariants $P_{k-1}\sigma_{kl}$ ($l = k + 2,\ k + 3$), provided $k + 3 \leqq n$. But a_{n-1} and a_n, not so determined, are found from $P_{k-1}a_k$ ($k = n - 1,\ n$). Hence *a fundamental system of modular seminvariants of f_n, for $p > n$, is given by*

$$\begin{aligned} &a_0, \quad \sigma_{0l} && (l = 2, \cdots, n), \\ &P_{k-1}\sigma_{kl} && (k = 1, \cdots, n-2;\ l = k+2, \cdots, n), \\ &P_{n-2}a_{n-1}, \quad P_{n-1}a_n. && \end{aligned} \tag{36}$$

For $n = 2, 3, 4$, these are (24′), (32), (34), respectively, except for the difference of notation indicated above. For $n = 5$, we see that *a fundamental system of modular seminvariants of f_5, for $p > 5$, is*

$$\begin{aligned} &a_0, \quad S_2, \quad S_3, \quad S_4, \quad S_5, \quad P_0S_{13}, \quad P_0S_{14}, \quad P_0S_{15}, \\ &P_1S_{24}, \quad P_1S_{25}, \quad P_2S_{35}, \quad P_3a_4, \quad P_4a_5, \end{aligned} \tag{37}$$

multiples of these four, any seminvariant can be reduced to $cA_0 + dA_0A_1$, which is an invariant only when identically zero. Hence 1, A_1, $A_0A_1A_2$ *and* I *form a complete set of linearly independent invariants of* F_2 *modulo* 2.

Next, let $p > 2$. The discriminant of f_2 is $D = S_2$. Any polynomial in the four fundamental seminvariants (24′) is a linear function of

$$a_0{}^iD^j, \quad P_0a_1{}^i, \quad P_1a_2{}^i \quad (i, j = 0, 1, \cdots, p-1),$$

since the product of P_0a_1 or P_1a_2 by a_0 is zero, that of P_1a_2 by P_0a_1 or D is zero, while $DP_0a_1 \equiv - Pa_1{}^3$. Further,

$$P_0 = 1 - a_0{}^{p-1}, \quad P_0[D^j - (-a_1{}^2)^j] \equiv 0,$$
$$P_1 = P_0 - P_0a_1{}^{p-1}, \quad a_0{}^{p-1}D^j \equiv D^j - (-1)^jP_0a_1{}^{2j},$$

modulo p. Hence any seminvariant is a linear function of

$$(40) \qquad \begin{array}{ll} a_0{}^{p-1}, \quad a_0{}^iD^j & (i = 0, 1, \cdots, p-2; j = 0, 1, \cdots, p-1), \\ \qquad P_0a_1{}^k, \quad P_1a_2{}^k & (k = 1, \cdots, p-1). \end{array}$$

The number of these is $p^2 + p - 1$. Hence (§ 13) *they form a complete set of linearly independent modular seminvariants of* f_2 *for* $p > 2$.

The invariant $A = A_1$ in § 6 of Lecture I becomes for two variables

$$(41) \qquad A = \{a_0{}^\mu + a_2{}^\mu(1 - a_0{}^{p-1})\}(1 - D^{p-1}) = a_0{}^\mu(1 - D^{p-1}) + P_1a_2{}^\mu,$$

where $\mu = (p-1)/2$. By the expansion of D^{p-1}, we get*

$$(42) \qquad A = (a_0{}^\mu + a_2{}^\mu)\left(1 - \sum_{i=0}^{\mu} a_0{}^ia_2{}^ia_1{}^{2\mu-2i}\right).$$

* *Transactions of the American Mathematical Society*, vol. 10 (1909), p. 132. To give a direct proof of the identity of the final expression (41) and (42), note that the product of the final factor in (42) by D equals $a_0a_2 - (a_0a_2)^{\mu+1}$ algebraically, so that the product AD is divisible by p. But the product of (41) by D is evidently divisible by p. It therefore remains only to treat the case $D \equiv 0$. Replacing $a_1{}^2$ by a_0a_2, we see that the final factor in (42) becomes $1 - (\mu + 1)a_0{}^\mu a_2{}^\mu$. Hence (41) and (42) are now identical if

$$a_0{}^\mu a_2{}^\mu(a_0{}^\mu - a_2{}^\mu) \equiv 0 \qquad (\text{mod } p).$$

But, if $a_0a_2 \neq 0$, $a_0{}^\mu a_2{}^\mu \equiv a_1{}^{2\mu} \equiv 1$, $a_0{}^\mu \equiv a_2{}^\mu \equiv \pm 1$.

Again, to find a fundamental system of f_4 for $p > 3$, we take a_0, S_2, S_3, S_4 and the products of P_0 by the functions $\frac{4}{3}S_{13}$ and $16S_{14}$ obtained from S_2 and S_3 by replacing a_0, a_1, a_2, a_3 by $4a_1$, $\frac{1}{3} \cdot 6a_2$, $\frac{1}{3} \cdot 4a_3$, a_4; then the product of P_1 by the function $2S_{24}$ obtained from S_2 by replacing a_0, a_1, a_2 by $6a_2$, $\frac{1}{2} \cdot 4a_3$, a_4; then P_2a_3 and P_3a_4, to characterize $P_2(4a_3x + a_4y)$. We again have (34).

13. *Number of Linearly Independent Seminvariants.*—Let $p > n$ and employ the notations of §11. In the classes C_k $(k < n)$. $A_{kk} = a_k \binom{n}{k}$ has $p - 1$ values, $A_{kk+1} = 0$, while $A_{kk+2}, \cdots, A_{kn}$ take independently the values $0, 1, \cdots, p - 1$. In the classes C_n, a_n has p values. Hence there are

$$p + \sum_{k=0}^{n-1} (p - 1)p^{n-k-1} = p + p^n - 1$$

distinct classes of forms f_n. Thus by § 11 of Lecture I, *there are exactly $p^n + p - 1$ linearly independent modular seminvariants of f_n when $p > n$.*

Derivation* of Modular Invariants from Seminvariants, §§ 14–15

14. *Invariants of the Binary Quadratic Form.*—First, let $p=2$. Any polynomial in the seminvariants (27) is a linear function of

$$1, \quad A_0, \quad A_1, \quad A_0A_1, \quad J, \quad A_0J \equiv A_0A_1A_2,$$

since $(A_0 + A_1)J \equiv 0$. Since there were six classes, these six seminvariants form a complete set of linearly independent seminvariants. Now a seminvariant is an invariant if and only if it is symmetrical in A_0 and A_2. But

$$I=(1-A_0)(1-A_1)(1-A_2)\equiv(1-A_0)(J+1+A_1) \qquad (\text{mod } 2).$$

Thus 1, A_1, A_0J and I are invariants. By subtracting constant

* While this method is usually longer than the method of Lecture I, it requires no artifices and makes no use of the technical theory of numbers. Moreover, it leads to the actual expressions of the invariants in terms of the seminvariants of a fundamental system, thus yielding material of value in the construction of covariants.

In place of the fourth and third we may evidently use

$$\lambda = (1 - A_1^2)A_2, \quad \sigma = A_1A_3 + A_0A_2 - A_1^2A_2^2 = t + A_0^2 + \lambda^2.$$

Here σ is the discriminant of F_3 for $p = 3$. By § 13 there are 11 classes of forms $\overline{F}_2$. Hence, by § 8, there are $3 \cdot 11$ classes of forms F_3. Thus there are exactly 33 linearly independent seminvariants of F_3. Since

$$A_1\lambda \equiv A_1\mu \equiv 0, \quad \sigma\lambda \equiv A_0\lambda^2, \quad \mu(\sigma + A_0^2) \equiv 0,$$
$$\mu(\lambda + A_0) \equiv 0, \quad (1 - A_1^2)\sigma \equiv A_0\lambda,$$

modulo 3, any polynomial in the seminvariants A_0, A_1, σ, λ, μ of the fundamental system is congruent to a linear function of

$$(44) \quad A_0^iA_1^j,\ A_0^i\sigma^k,\ A_0^iA_1\sigma^k,\ A_0^i\lambda^k,\ A_0^i\mu^k \quad (i, j=0, 1, 2;\ k=1, 2).$$

Hence these 33 functions form a complete set of linearly independent seminvariants of F_3. The seminvariants

$$P = 1 - A_1^2 - \lambda^2 = (1 - A_1^2)(1 - A_2^2),$$
$$(45) \quad I_0 = (1 - A_0^2)(P - \mu^2) = \prod_{i=0}^{3}(1 - A_i^2),$$
$$E = A_0A_1(\sigma - \sigma^2) + A_0\mu = A_0A_3(A_0A_2 - A_1A_3 + A_1^2 - A_2^2)$$

are seen to be invariants as follows.* The weights of the terms of each are all even or all odd. Moreover, under the substitution $(A_0A_3)(A_1A_2)$, induced upon the coefficients of F_3 by the interchange of x and y, the functions σ, P and I_0 are unaltered, while E is changed in sign. Hence σ, P, I_0 are absolute invariants, while E is an invariant of index unity. We now have 7 linearly independent invariants

$$(46) \qquad I_0, \quad E, \quad E^2, \quad \sigma, \quad \sigma^2, \quad P, \quad 1.$$

Noting that

$$(47) \qquad E^2 = A_0^2\mu^2 + A_0^2(\sigma - \sigma^2 + \lambda^2) - A_0\lambda,$$

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* Or by general theorems, *Transactions of the American Mathematical Society*, vol. 8 (1907), pp. 206–207. Note that E is the eliminant of $F_3 \equiv 0$, $x^3 \equiv x$, $y^3 \equiv y$ (mod 3).

Since (42) is therefore a seminvariant and is symmetrical in a_0 and a_2 and since the weight of every term is divisible by $p-1$, A is an absolute invariant. By (41),

$$(43)\qquad \begin{aligned} &A^2 \equiv a_0^{2\mu}(1-D^{p-1}) + P_1 a_2^{2\mu}, \quad (1-a_0^{p-1})D^{p-1} \equiv P_0 a_1^{p-1},\\ &A^2 + D^{p-1} - 1 \equiv -I_0, \quad I_0 = (1-a_0^{p-1})(1-a_1^{p-1})(1-a_2^{p-1}). \end{aligned}$$

Hence also I_0 is an absolute invariant. Subtracting multiples of

$$I_0 = 1 - a_0^{p-1} - P_0 a_1^{p-1} - P_1 a_2^{p-1}, \quad A, \quad D^j \quad (j=0, 1, \cdots, p-1),$$

we may reduce any seminvariant to a linear function of the expressions (40) other than $P_1 a_2^{p-1}$, $P_1 a_2^{\mu}$, D^j $(j = 0, \cdots, p-1)$. The resulting linear function L is not an invariant. For example, if $p = 3$, it is

$$L = aa_0^2 + ba_0 + ca_0D + da_0D^2 + eP_0a_1 + fP_0a_1^2 \quad (a, \cdots, f \text{ constants}).$$

Interchange a_0 and a_2, and change the sign of a_1. We get

$$aa_2^2 + ba_2 + ca_2D + da_2D^2 + (1-a_2^2)(fa_1^2 - ea_1).$$

This is to be identically congruent to the invariant L. Taking $a_2 = 0$, we see that $e = f = a = b = 0$, $c = d$. Then $L = ca_0a_2(a_0 + a_2) + ca_0^2a_1^2a_2$ is not symmetric in a_0 and a_2. Hence $L \equiv 0$. For any p, a like result may be proved by considering separately the terms of L of constant weights modulo $p-1$. Hence in accord with § 11 of Lecture I, *a complete set of linearly independent invariants of f_2, for $p > 2$, is given by I_0, A and the powers of D.* In place of $D^0 = 1$, we may use A^2, in view of (43).

15. *Invariants of the Binary Cubic Modulo* 3.—A fundamental system of seminvariants of F_3 modulo 3 is given by A_0 and a fundamental system of

$$\bar{F}_2 = A_1x^2 + (A_0 + A_2)xy + A_3y^2.$$

Hence, by (24), a fundamental system for F_3 is given by

$$A_0, \quad A_1, \quad t = A_1A_3 - (A_0 + A_2)^2, \quad (1 - A_1^2)(A_0 + A_2),$$
$$\mu = (1 - A_1^2)[1 - (A_0 + A_2)^2]A_3.$$

LECTURE III

INVARIANTS OF A MODULAR GROUP. FORMAL INVARIANTS AND COVARIANTS OF MODULAR FORMS. APPLICATIONS

Invariants of Certain Modular Groups, §§ 1–4

1. *Introduction.*—Let G be any given group of g linear homogeneous transformations on the indeterminates $x_1, \cdots, x_m$ with integral coefficients taken modulo p, a prime. Hurwitz* raised the question of the existence of a finite fundamental system of invariants of G. For the relatively unimportant case in which g is not divisible by p, he readily obtained an affirmative answer by use of Hilbert's well known theorem on a set of homogeneous functions, but emphasized the difficulty of the problem in the general case.

In § 5 I shall consider the relation of this question to that of modular covariants and formal invariants of a system of forms and incidentally answer the above question for special groups of orders divisible by p.

I shall, however, first present a simplification of my own work on the total group. Its invariants are universal covariants, i. e., covariants of any system of modular forms (§ 13). It was from the latter standpoint that I was led to the subject of invariants of a modular group independently of Hurwitz's paper, in the title of which the word invariant does not occur.

2. *Invariants of the Total Binary Group.*—Consider the group G of all modular linear homogeneous transformations with integral coefficients of determinant unity:

$$(1) \quad x' \equiv bx + dy, \quad y' \equiv cx + ey, \quad be - cd \equiv 1 \qquad (\text{mod } p).$$

The term point will be used in the sense of homogeneous coordinates, so that $(x, y) = (ax, ay)$, while $(0, 0)$ is excluded.

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* *Archiv der Mathematik und Physik,* (3), vol. 5 (1903), p. 25.

we may employ the functions (46) to delete from (44)

$$\mu^2, \quad A_0\mu, \quad A_0^2\mu^2, \quad \sigma, \quad \sigma^2, \quad \lambda^2, \quad 1$$

in turn (no one of these terms being reintroduced at a later stage). There remain 11 seminvariants of odd weight

$$(48) \qquad A_0^iA_1, \quad A_0^iA_1\sigma, \quad A_0^iA_1\sigma^2, \quad \mu, \quad A_0^2\mu \qquad (i = 0, 1, 2),$$

and 15 of even weight

$$(49) \quad A_0, A_0^2, A_0^iA_1^2, A_0\sigma, A_0\sigma^2, A_0^2\sigma, A_0^2\sigma^2, A_0^i\lambda, A_0\lambda^2, A_0^2\lambda^2, A_0\mu^2.$$

Now the weight and index of a seminvariant of F_3 modulo 3 are both even or both odd.* A linear combination of the functions (48) which is changed in sign by the substitution $(A_0A_3)(A_1A_2)$ is seen to be identically zero (it suffices to set $A_3 = 0$, $A_2 = 0$ in turn). A linear combination of the functions (49) which is unaltered by that substitution is seen similarly to be identically zero. Hence† *a complete set of linearly independent invariants of F_3 modulo 3 is given by* (46).

* When the sign of y is changed, a seminvariant is unaltered or changed in sign according as its weight is even or odd.

† Another proof, using the classes of F_3 under the group of all binary linear transformations of determinant unity modulo 3, and involving a use of more technical theory of numbers, is given in *Transactions of the American Mathematical Society*, vol. 10 (1909), pp. 149–154. The case of any modulus p is there treated.

Hence only real special points are invariant under a transformation [other than (3)] whose characteristic congruence has an integral root. Moreover, all real points are conjugate under the group G. Indeed,

$$x' \equiv bx, \quad y' \equiv x + b^{-1}y, \quad \text{and} \quad x' \equiv -y, \quad y' \equiv x$$

replace (1, 0) by (b, 1) and (0, 1) respectively. Hence if an invariant of G vanishes for one of the real points, it vanishes for all and has the factor

$$(6) \qquad L = y \prod_{a=0}^{p-1} (x - ay) \equiv x^p y - xy^p \qquad (\text{mod } p),$$

the congruence following from Fermat's theorem. Obviously, any transformation of G replaces a real point by a real point, and therefore L by kL. The constant k is in fact unity and L is an invariant of G. Indeed, for

$$(7) \qquad x \equiv aX + bY, \quad y \equiv cX + dY \qquad (\text{mod } p),$$

where $a, \cdots, d$ are integers of determinant $\Delta = ad - bc$,

$$(8) \quad \begin{vmatrix} x^p & y^p \\ x & y \end{vmatrix} \equiv \begin{vmatrix} aX^p + bY^p & cX^p + dY^p \\ aX + bY & cX + dY \end{vmatrix} = \Delta \begin{vmatrix} X^p & Y^p \\ X & Y \end{vmatrix} \qquad (\text{mod } p).$$

Next, suppose that (5) has no integral root and therefore two Galois imaginary roots. By (2), each root ρ uniquely determines a point (x, y) with $y \neq 0$. We may therefore take $y = 1$, whence $cx \equiv \rho - e$. The resulting two special points are therefore imaginary points of the form ($r\rho + s$, 1), where r and s are integers modulo p, and r is not divisible by p. The imaginaries introduced* by new transformations are expressible linearly in terms of this ρ. Indeed, $(2\rho - \alpha)^2 \equiv A$, where $A = \alpha^2 - 4$ is a quadratic non-residue of p (i. e., is not the remainder when the square of any integer is divided by p). Thus $A \equiv a^2\nu$, where ν is a fixed non-residue of p. Hence the roots of all congruences (5) having no integral roots are expressible in the form $k + l\sqrt{\nu}$, where k and l are integers.

* There are no new ones if $p = 2$, since $\alpha \equiv 0$ (mod 2).

We do not restrict the coordinates to be integers, but permit their ratio to be a root of any congruence with integral coefficients modulo p. A point is called *real* if the ratio of its coordinates is rational.

A point (x, y) is invariant under a transformation (1) if $x' \equiv \rho x$, $y' \equiv \rho y$, or

$$(2) \quad (b-\rho)x + dy \equiv 0, \quad cx + (e-\rho)y \equiv 0 \qquad (\text{mod } p).$$

If these congruences hold identically as to x, y, then

$$d \equiv c \equiv 0, \quad b \equiv e \equiv \pm 1 \qquad (\text{mod } p)$$

and the transformation is one of the transformations

$$(3) \qquad x' \equiv \pm x, \quad y' \equiv \pm y \qquad (\text{mod } p),$$

which leave every point invariant.

A *special* point is one invariant under at least one transformation (1) not of the form (3). There are $p(p^2-1)$ transformations (1). We shall assume in the text that $p > 2$ (relegating to foot-notes the modifications to be made when $p = 2$). Then there are two transformations (3). Hence any non-special point is one of exactly*

$$(4) \qquad \omega = \tfrac{1}{2}p(p^2-1)$$

conjugate points under the group G, while a special point is one of fewer than ω conjugates.

Let (x, y) be a special point and let (1) be a transformation, not of the form (3), which leaves it invariant. Thus the congruences (2) are not both identities. The determinant of their coefficients must therefore be divisible by p. Hence ρ is a root of the *characteristic* congruence (in which $\alpha = b + e$)

$$(5) \qquad \rho^2 - \alpha\rho + 1 \equiv 0 \qquad (\text{mod } p).$$

First, suppose that (5) has an integral root ρ. For this value of ρ, one of the congruences (2) is a consequence of the other, and the ratio $x : y$ is uniquely determined as an integer modulo p.

* For $p = 2$, ω is to be replaced by $2(2^2-1) = 6$.

integral root], and consequently a value which is a quadratic non-residue of p. Then, by choice of c, q can be made congruent to any assigned non-residue.

Having made $q \equiv i \pmod{p}$ by choice of c and e, we proceed to choose integral solutions b and d of (9) such that N will be congruent to any assigned integer j. If $c \equiv 0$, so that $e \not\equiv 0$, we take $d \equiv j/e$. If $c \not\equiv 0$, we eliminate d from N by use of (9) and obtain

$$N \equiv \frac{1}{c}(bq - e - c\alpha), \quad q = c^2 + \alpha ce + e^2.$$

Since $q \not\equiv 0$, we may make $N \equiv j$ by choice of b.

We have therefore proved that there are exactly $p^2 - p$ imaginary special points, viz., $(r\rho + s, 1)$, $r \not\equiv 0$, and that they are all conjugate under the group G. Hence any invariant of G which vanishes for an imaginary special point has the factor

$$Q = \frac{x^{p^2}y - xy^{p^2}}{L} = \frac{x^{p^2-1} - y^{p^2-1}}{x^{p-1} - y^{p-1}}. \tag{10}$$

Indeed, the numerator of the first fraction vanishes for $x = r\rho + s$, $y = 1$, since

$$(r\rho + s)^{p^2} \equiv r\rho^{p^2} + s, \quad \rho^{p^2} \equiv \rho \pmod{p},$$

the last congruence* being a case of Galois's generalization of Fermat's theorem. We have divided out L, which vanishes for the real points $(s, 1)$ and $(1, 0)$. Since any transformation of G replaces one of our imaginary points by another, it replaces Q by kQ. The constant k is in fact unity and Q is an invariant of G. Indeed, (8) holds if we replace the exponents p by p^2. Hence the quotient Q is invariant† under all transformations (7).

* It may be proved by noting that (5) implies

$$(\rho^2 - \alpha\rho + 1)^p \equiv \rho^{2p} - \alpha\rho^p + 1 \equiv 0 \pmod{p},$$

so that ρ^p is the second root of (5). By the same argument, $(\rho^p)^p$ is a root, distinct from ρ^p, and hence identical with ρ.

† I gave the notation Q to the invariant (10) since it is the product of all of the binary quadratic forms $x^2 + \cdots$ which are irreducible modulo p. Indeed, the latter vanishes for two points of the form $(r\rho + s, 1)$ and $(r\rho' + s, 1)$, where ρ and ρ' are the roots of (5) and r, s are integers, $r \not\equiv 0$, and conversely.

Hence the special points invariant under transformations whose characteristic congruences have no integral roots are all of the form $(r\rho + s, 1)$, where r and s are integers, r not divisible by p, while ρ is a fixed root of a particular one of these congruences (5).

We next show that these $p^2 - p$ imaginary special points are all conjugate under the group G. It suffices to prove that they are all conjugate with $(\rho, 1)$, which is invariant under

$$x' \equiv \alpha x - y, \quad y' \equiv x.$$

Now transformation (1) replaces $(\rho, 1)$ by $(R, 1)$, where

$$R = \frac{b\rho + d}{c\rho + e}.$$

We are to prove that there exist integers b, c, d, e satisfying

$$(9) \qquad be - cd \equiv 1 \pmod{p},$$

such that $R \equiv r\rho + s$, where r and s are any assigned integers for which r is not divisible by p. Denote the second root of (5) by ρ' and multiply the numerator and denominator of R by $c\rho' + e$. Using (9), we get

$$R \equiv \frac{\rho + N}{q}, \quad N = bc + de + dc\alpha, \quad q = c^2 + \alpha ce + e^2.$$

We first show* that we can choose integers c and e such that $q \equiv i \pmod{p}$, where i is any assigned integer not divisible by p. If i is a quadratic residue of p, we may take $c = 0$. Next, let i be a quadratic non-residue of p. Taking $c \not\equiv 0$, $e \equiv kc$, we have

$$q \equiv c^2 f(k), \quad f(k) = 1 + \alpha k + k^2.$$

Now $f(k) \equiv f(K)$ if and only if $K \equiv k$ or $K \equiv -\alpha - k$. Hence the $p - 1$ values of k other than $-\alpha/2$ give by pairs the same value of $f(k)$. Thus for $k = 0, \cdots, p - 1$, $f(k)$ takes $1 + \frac{1}{2}(p-1)$ incongruent values, no one a multiple of p [since (5) has no

* If $p = 2$, then $\alpha \equiv 0$; taking $c = 1$, $e = 0$, we have $q \equiv 1 \equiv i \pmod{2}$.

of the group composed of the p powers of the transformation

$$(13)\qquad x' \equiv x + y, \quad y' \equiv y \qquad (\mathit{mod}\ p)$$

is given by y and λ, where

$$(14)\quad \lambda = x(x+y)(x+2y)\cdots(x+\overline{p-1}y) \equiv x^p - xy^{p-1} \qquad (\mathit{mod}\ p).$$

Now (1, 0) is the only special point, being the only point unaltered by (13) or its kth power, $k < p$. Hence an invariant not having a factor y or λ vanishes at imaginary points falling into sets each of p points conjugate under our group. As at the end of § 2, the invariant is a product of factors $y^p + \tau\lambda$ so related that the product equals a polynomial in y^p and λ with integral coefficients.

Other results will be merely stated, since they are not presupposed in what follows. Within the group G of all transformations (1), any subgroup of order a multiple of p is conjugate with one containing (13) and transformations exclusively of the form

$$(15)\qquad x' \equiv tx + ly, \quad y' \equiv t^{-1}y \qquad (\mathrm{mod}\ p),$$

and having y and λ as a fundamental system of invariants.* The invariants of any subgroup whose order is prime to p have been found.†

4. *Invariants of the Total Group on m Variables.*—The functions

$$(16)\quad L_m = \begin{vmatrix} x_1^{p^{m-1}} & \cdots & x_m^{p^{m-1}} \\ x_1^{p^{m-2}} & \cdots & x_m^{p^{m-2}} \\ \cdot & \cdots & \cdot \\ x_1^{p} & \cdots & x_m^{p} \\ x_1 & \cdots & x_m \end{vmatrix}, \quad Q_{ms} = \begin{vmatrix} x_1^{p^m} & \cdots & x_m^{p^m} \\ \cdot & \cdots & \cdot \\ x_1^{p^{s+1}} & \cdots & x_m^{p^{s+1}} \\ x_1^{p^{s-1}} & \cdots & x_m^{p^{s-1}} \\ \cdot & \cdots & \cdot \\ x & \cdots & x_m \end{vmatrix} \div L_m$$

are seen, by a generalization of (8), to be invariants of index 1 and 0 respectively of the group Γ_m of all linear homogeneous transformations on $x_1, \cdots, x_m$ with integral coefficients modulo p.

* *Bulletin of the American Mathematical Society*, vol. 20 (1913), pp. 132–4.
† *American Journal of Mathematics*, vol. 33 (1911), p. 175.

We are now ready to prove that *any rational integral invariant I, with integral coefficients, of the group G is a rational integral function of L and Q with integral coefficients.*

After removing possible factors L and Q, we may assume that I vanishes for no special point. If I is not a constant, it vanishes at a point (c, d) and hence at the ω distinct points conjugate with (c, d) under the group G. The invariants*

$$(11) \qquad q = Q^{\frac{p+1}{2}}, \quad l = L^{\frac{p(p-1)}{2}}$$

are of degree ω. The constant τ, determined by

$$q(c, d) + \tau \cdot l(c, d) \equiv 0 \qquad (\mathrm{mod}\ p),$$

is a root of a congruence of a certain degree t with integral coefficients and irreducible modulo p. Now $q + \tau l$ is a factor of I. Since q, l and I have integral coefficients, I has also the factors

$$(12) \qquad q + \tau^{p}l, \quad q + \tau^{p^2}l, \quad \cdots, \quad q + \tau^{p^{t-1}}l.$$

For, by Galois's theorem mentioned above,

$$\tau, \quad \tau^{p}, \quad \tau^{p^2}, \quad \cdots, \quad \tau^{p^{t-1}}$$

are the roots of our irreducible congruence of degree t. Since the conditions which imply that $q + zl$ shall be a factor of I are congruences satisfied when $z = \tau$, they are satisfied when $z = \tau^{p^k}$. Hence if we multiply $q + \tau l$ by the product of the invariants (12), we obtain an invariant T with integral coefficients modulo p. Since L and Q have no common factor, no two of the functions $q + \tau l$ and (12) have a common factor. Hence T is a factor of I. Proceeding in like manner with I/T, we arrive finally at the truth of the theorem.†

3. *Invariants of Smaller Binary Groups.*—We shall later need the theorem that *a fundamental system of rational integral invariants*

* If $p = 2$, we omit the divisor 2 in the exponents.

† Proved less simply in *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 1. Still simpler is the proof that various coefficients of an invariant are zero, *Quarterly Journal of Mathematics*, 1911, p. 158.

tion (7) if

$$(19) \qquad P(A_0, A_1, \cdots, A_r) \equiv \Delta^\lambda P(a_0, a_1, \cdots, a_r) \pmod{p},$$

identically as to $a_0, \cdots, a_r$, after the A's have been replaced by their values (18) in terms of the a_i. If P is invariant modulo p under all transformations (7), it is called a formal invariant modulo p of f.

The term formal is here used in connection with a form f whose coefficients are arbitrary variables in contrast to the case, treated in the earlier Lectures, in which the coefficients are undetermined integers taken modulo p. In the latter case, (19) necessarily becomes an identical congruence in the a's only after the exponent of each a is reduced to a value less than p by means of Fermat's theorem $a^p \equiv a \pmod{p}$.

The functions (18) are linear in $a_0, \cdots, a_r$. It is customary to say that relations (18) define a linear transformation on $a_0, \cdots, a_r$ which is induced by the binary transformation (7). Let Γ be the group of all of the transformations (18) induced by the group of all of the binary transformations (7). Making no further use of the form f, we may state the above problem of the determination of the formal invariants of f in the following terms. We desire a fundamental system of invariants of group Γ. This problem is of the type proposed in § 1; the group Γ is a special group of order a multiple of p. Here and below the term invariant is restricted to rational integral functions of $a_0, \cdots, a_r$.

A theory of formal invariants has not been found. For no form f has a fundamental system of formal invariants been published. Some light is thrown upon this interesting but difficult problem by the following complete treatment of a binary quadratic form, first for the exceptional case $p = 2$ and next for the case $p > 2$, and preliminary treatment of a binary cubic form.

6. *Formal Invariants Modulo 2 of a Binary Quadratic Form.*—Write

$$(20) \qquad f = ax^2 + bxy + cy^2,$$

Since L_m is an invariant of Γ_m and has the factor x_1, it follows from an examination of its diagonal term that*

$$(17)\quad L_m \equiv \prod_{k=1}^{m} \sum_{c_i=0}^{p-1} (x_k + c_{k+1}x_{k+1} + \cdots + c_m x_m) \qquad (\text{mod } p),$$

in which occurs one of each set of proportional linear forms modulo p. A like proof shows that the numerator of Q_{ms} is divisible by each of the linear functions (17) and hence by L_m, modulo p.

Making use of the theorem in § 2, I have proved by induction† that the m invariants L_m, Q_{m1}, $\cdots$, Q_{mm-1} are independent and form a fundamental system of rational integral invariants of Γ_m.

A fundamental system of invariants of the group of all modular linear transformations on two sets of two cogredient variables has been obtained very recently by Dr. W. C. Krathwohl in his Chicago dissertation.‡

Formal Invariants and Seminvariants of Modular Forms, §§ 5–13

5. *Formal Modular Invariants.*—Consider a binary form

$$f(x, y) = a_0x^r + a_1x^{r-1}y + \cdots + a_ry^r,$$

in which x, y, a_0, $\cdots$, a_r are arbitrary variables. The transformation (7) with integral coefficients, whose determinant Δ is not divisible by the prime p, replaces f by a form

$$\phi(X, Y) = A_0X^r + A_1X^{r-1}Y + \cdots + A_rY^r,$$

in which

$$(18)\quad A_0 = f(a, c), \quad A_1 = ra^{r-1}ba_0 + \cdots, \quad \cdots, \quad A_r = f(b, d).$$

A polynomial $P(a_0, \cdots, a_r)$ with integral coefficients is called a formal invariant modulo p of index λ of f under the transforma-

* E. H. Moore, *Bulletin of the American Mathematical Society*, vol. 2 (1896), p. 189. His proofs do not use the invariantive property. A like remark is true of the proof that the product (17), in the case $x_m = 1$, is congruent to a determinant of order $m - 1$, then obviously equal to L_m, by R. Levavasseur, *Mémoires de l'Académie des Sciences de Toulouse*, ser. 10, vol. 3 (1903), pp. 39–48; *Comptes Rendus*, 135 (1902), p. 949.

† *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 75.

‡ *American Journal of Mathematics*, October, 1914.

Evident formal seminvariants are a, $\Delta = b^2 - ac$, and

$$\beta = \prod_{t=0}^{p-1} (ta + b) \equiv b^p - ba^{p-1} \pmod{p}, \tag{27}$$

$$\gamma_k = \prod_{t=0}^{p-1} \{(t^2 - k)a + 2tb + c\} \quad (k = 0, 1, \cdots, p - 1). \tag{28}$$

Indeed, the linear function under the product sign in (28) is transformed by (26) into the function derived from it by replacing t by $t + 1$. As in (27),

$$[\gamma_k]_{a=0} \equiv c^p - cb^{p-1} \pmod{p}. \tag{29}$$

Let $S(a, b, c)$ be a homogeneous rational integral seminvariant with integral coefficients. Then, by (26),

$$S(0, b, c) \equiv S(0, b, 2b + c) \pmod{p}.$$

Thus, by § 3, $S(0, b, c)$ equals a polynomial in b, $c^p - cb^{p-1}$. Hence, by (29),

$$S(a, b, c) \equiv a\sigma(a, b, c) + \phi(b, \gamma_k) \pmod{p},$$

where σ and ϕ are polynomials in their arguments. Now

$$b^{2i} = \Delta^i + a(\quad), \quad b^{p+2i} = \beta\Delta^i + a(\quad).$$

Hence

$$S = a\lambda(a, b, c) + \psi(\beta, \Delta, \gamma_k) + \sum_{i=0}^{(p-3)/2} d_i b^{2i+1} \gamma_k^{e_i}, \tag{30}$$

where λ and ψ are polynomials in their arguments, and d_i is an integer.

When y is multiplied by a primitive root ρ of p, a, b, c are multiplied by 1, ρ, ρ^2, respectively. Hence β is multiplied by ρ, while, by (29), γ_k and Δ are multiplied by ρ^2. If therefore we attribute the weights 0, 1, 2 to a, b, c, respectively, and the weight $s + 2t$ to $a^r b^s c^t$, we see that the weight of every term of γ_k is congruent to 2 modulo $p - 1$.

We can now prove that every d_i is divisible by p. For, if not, the seminvariant $S - \psi$ has a term of odd weight, so that every

where a, b, c are arbitrary variables. Under the transformation

$$x = x' + y', \quad y = y', \tag{21}$$

f becomes f', in which the coefficients are

$$a' \equiv a, \quad b' \equiv b, \quad c' \equiv a + b + c \qquad (\text{mod } 2). \tag{22}$$

By § 3, the only invariants under $d' \equiv d$, $c' \equiv c + d$, modulo 2, are the polynomials in d and $c(c + d)$. Take $d = a + b$. Hence *the only seminvariants of f are the polynomials in a, b and*

$$s = c(c + a + b). \tag{23}$$

Such a polynomial is an invariant of f if and only if it is unaltered by the substitution (ac) induced by (xy). Thus

$$b, \quad k = as, \quad q = b(a + c) + a^2 + ac + c^2 = s + ab + a^2 \tag{24}$$

are invariants of f. Introducing q in place of s, we see that any seminvariant is a polynomial in a, b, q. Consider an invariant of this type. Since its terms free of a are invariants, the sum of its terms involving a is an invariant with the factor a and hence also the factors c and $a + b + c$, the last by (22). Hence this sum has the factor k, and its quotient by k is an invariant. By induction we have the theorem:

Any rational integral formal invariant of f equals a rational integral function of b, q, k.*

7. *Formal Seminvariants of a Binary Quadratic Form for $p > 2$.* Write

$$f = ax^2 + 2bxy + cy^2, \tag{25}$$

where a, b, c are arbitrary variables. Under the transformation (21), f becomes f', whose coefficients are

$$a' = a, \quad b' = a + b, \quad c' = a + 2b + c. \tag{26}$$

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* Replace x_1, x_2, x_3, of § 4 by a, b, c; then

$$L_3 = bk(k + bq), \quad Q_{32} = b^4 + bk + q^2, \quad Q_{31} = b^2q^2 + bqk + b^3k + k^2.$$

determinant unity. It suffices to prove that this seminvariant is unaltered by the substitution

$$a' = c, \quad c' = a, \quad b' = -b, \tag{33}$$

induced by the transformation $x = y'$, $y = -x'$. Under (33), the general factor in (28) is replaced by

$$(t^2 - k)\{(T^2 - K)a + 2Tb + c\},$$

where

$$T = \frac{-t}{t^2 - k}, \qquad K = \frac{k}{(t^2 - k)^2}.$$

Hence K is quadratic non-residue of p when k is. Also,

$$\prod_{t=0}^{p-1}(t^2 - k) = -k\left\{\prod_{t=1}^{(p-1)/2}(k - t^2)\right\}^2 \equiv -k(k^{\frac{p-1}{2}} - 1)^2 \equiv -4k \pmod{p}$$

if k is a non-residue. To show that the product of the resulting numbers $-4k$ is congruent to unity, we set $x = 0$ in

$$\prod_k (x - k) \equiv x^{\frac{p-1}{2}} + 1 \pmod{p}, \tag{34}$$

and note that $2^{p-1} \equiv 1$. Hence (32) is unaltered by (33) and is an absolute invariant of f under G.

It is very easy to verify that

$$J = a\gamma_0 \tag{35}$$

is unaltered by (33), so that J is an invariant of f under G.

If an invariant has the factor β, it has the factor

(36) $B = \beta \Pi \gamma_r$ (r ranging over the quadratic residues of p).

For, under the substitution (33), $b + \tau a$ ($\tau \neq 0$) becomes $\tau(c - b/\tau)$. By choice of τ, we reach $c + 2tb$, where t is any assigned integer not divisible by p. This is a factor of γ_k where $k \equiv t^2$.

The fact that B is an invariant may be verified as in the case of (32) or deduced from the fact that

$$a\beta \prod_{k=0}^{p-1} \gamma_k = a\gamma_0 \cdot B\Gamma$$

term of λ is of odd weight and hence has the factor b. Thus $S - \psi$ has the factor b and therefore the factor β, so that its terms free of a have the factor b^p. But this is impossible, since $2i + 1 < p$ and (29) does not have the factor b.

Hence $S - \psi$ has the factor a and the quotient is a seminvariant of the form $a\lambda' + \psi'$. Proceeding in this way, we obtain the theorem:

Any seminvariant is a polynomial in a, Δ, β and any single γ_k.

Of these, β alone is of odd weight. Hence any seminvariant is a polynomial in $a, \Delta, \gamma_k, \beta^2$ or the product of such a polynomial by β. But

$$(31) \qquad \beta^2 \equiv a^p\gamma_0 + \Delta(\Delta^{\frac{p-1}{2}} - a^{p-1})^2 \qquad (\text{mod } p).$$

To prove this, it suffices to show that the second member is divisible by b and hence by β, and being of even weight therefore by β^2, and to remark that each member of (31) reduces to b^{2p} for $a = 0$. Now

$$[\gamma_0]_{b=0} = \prod_{t=0}^{p-1} (t^2a + c) = c\left\{\prod_{t=1}^{(p-1)/2} (t^2a + c)\right\}^2$$

$$\equiv c\{c^{\frac{p-1}{2}} - (-a)^{\frac{p-1}{2}}\}^2 \qquad (\text{mod } p),$$

$$a^p[\gamma_0]_{b=0} \equiv ac\{(-ac)^{\frac{p-1}{2}} - a^{p-1}\}^2 \qquad (\text{mod } p).$$

But Δ reduces to $-ac$ for $b = 0$. Hence the second member of (31) has the factor b. We therefore have the theorem:

For $p > 2$, any formal seminvariant of a binary quadratic form is a polynomial in a, Δ, γ_0 or the product of such a polynomial by β.

8. *Formal Invariants of a Binary Quadratic Form for $p > 2$.* The product

$$(32) \quad \Gamma = \prod_k \gamma_k \quad (k \text{ ranging over the quadratic non-residues of } p)$$

is an absolute invariant of f under the group G of all binary transformations with integral coefficients taken modulo p of

is involved in the assumption that a certain c_j is not divisible by p. First, the remaining c_i are divisible by p. For if also $c_i \not\equiv 0$, let $k_i\Delta^{r_i}\Gamma^{s_i}$ be the term of P_i of highest degree in Δ. Since γ_0 and Γ are of degrees p and np, and of weights $\equiv 2$ and 0 (mod $p-1$), $\gamma_0{}^iP_i$ is of degree $pi + 2r_i + s_inp$ and of weight $\equiv 2i + 2r_i$ (mod $p-1$). But $p \equiv 1$ (mod n). Hence

$$i + 2r_i \equiv j + 2r_j, \quad 2i + 2r_i \equiv 2j + 2r_j \qquad (\text{mod } n),$$

so that $i \equiv j$ (mod n). But i and j are positive integers $< n$. Hence $i = j$. Multiplying our invariant by a suitably chosen integer, we have the invariant

$$(39) \quad \gamma_0{}^jP_j(\Delta, \Gamma) + \sum_{i=0}^{n-1}\gamma_0{}^i\phi_i(a, \Delta, \Gamma), \quad P_j = \Delta^r\Gamma^s + \cdots.$$

Now $-(c - ka)b^{p-1}$ is the term of highest degree in b in γ_k. Hence

$$(40) \qquad \gamma_0 = -cb^{p-1} + \cdots, \quad \Gamma = \sigma b^{n(p-1)} + \cdots,$$

$$(41) \quad \sigma = \prod_k\{-(c - ka)\} \equiv (-c)^n + (-a)^n \qquad (\text{mod } p),$$

where k ranges over the non-residues of p, the last following from (34) for $x = c/a$. Since γ_0 and Γ are of even weights, only even powers of b enter (39). Hence an invariant (39) is symmetrical in a and c. We shall prove that this is not the case for the terms of highest degree in b. For $\gamma_0{}^jP_j$ this term is

$$(42) \qquad (-c)^j\sigma^s b^\beta, \quad \beta = j(p-1) + 2r + sn(p-1).$$

Let $C_ia^{e_i}\Delta^{f_i}\Gamma^{g_i}$ be one of the terms of ϕ_i in which the exponent of b is a maximum. Then in $\gamma_0{}^i\phi_i$ the highest power of b occurs in the terms

$$(43) \quad C_ia^{e_i}(-c)^i\sigma^{g_i}b^{\beta_i}, \quad \beta_i = 2f_i + g_in(p-1) + i(p-1).$$

Since the weight and degree is the same as for (42),

$$(44) \qquad \begin{aligned} 2i + \beta_i &\equiv 2j + \beta \qquad (\text{mod } p-1), \\ e_i + i + g_in + \beta_i &= j + sn + \beta. \end{aligned}$$

is an invariant, being the product of all non-proportional linear functions of a, b, c with integral coefficients modulo p.

Hence any invariant is the product of a power of B by an invariant which is a polynomial P in a, Δ, γ_0.

Since γ_k is a seminvariant not divisible by β, it equals a polynomial in a, Δ, γ_0 (§ 7). But if $a = 0$, $\gamma_k \equiv \gamma_0 \pmod{p}$, by (29), and $\Delta = b^2$ is free of c, so that γ_k is not a polynomial in a and Δ only. Hence

$$\gamma_k \equiv \gamma_0 + g_k(a, \Delta) \pmod{p}. \tag{37}$$

For $p = 3$, the polynomial P therefore equals a polynomial in a, Δ, $\gamma_2 = \Gamma$. Now an invariant $\phi(a, \Delta, \Gamma)$ differs from the invariant $\phi(0, \Delta, \Gamma)$ by an invariant with the factor a and hence the factor (35). Treating the quotient similarly, we ultimately obtain the following theorem for the case $p = 3$:

A fundamental system of formal invariants of the binary quadratic form f modulo p, $p > 2$, is given by the discriminant Δ and Γ, J, B, defined by (32), (35), (36). *The product of the last three is congruent modulo p to the product of all the non-proportional linear functions of the coefficients of f.*

To prove the theorem for $p > 3$, note first, by (37), that Γ, given by (32), differs from γ_0^n by a polynomial in γ_0, a, Δ of degree $n - 1$ in γ_0, where $n = (p - 1)/2$. Hence a polynomial in a, Δ, γ_0 equals a polynomial in $a, \Delta, \gamma_0, \Gamma$ of degree at most $n - 1$ in γ_0. Subtract from each the terms of the latter involving only the invariants Δ, Γ. We have therefore to investigate invariants of the type

$$\sum_{i=1}^{n-1} c_i\gamma_0^i P_i(\Delta, \Gamma) + \sum_{i=0}^{n-1} \gamma_0^i\phi_i(a, \Delta, \Gamma), \tag{38}$$

in which the c_i are integers, while P_i and ϕ_i are polynomials in their arguments, and ϕ_i has the factor a. If every $c_i \equiv 0$, the invariant has the factor a and hence the factor $a\gamma_0 = J$, and the quotient by J is an invariant which may be treated similarly. The theorem will therefore follow if we show that a contradiction

Hence α, β and γ_k, given by (27) and (28), are again seminvariants; also,

$$(47) \quad \delta_{jk} = \prod_{t=0}^{p-1} \{(t^3 - 3kt - j)a + 3(t^2 - k)b + 3tc + d\} \qquad (j, k = 0, \cdots, p-1).$$

Indeed, if $F_t(a, b, c, d)$ is the function in brackets,

$$F_t(a', b', c', d') = F_{t+1}(a, b, c, d).$$

Any invariant with the factor a has the factor

$$(48) \quad a\delta_{00} = a\prod_{t=0}^{p-1} (t^3a + 3t^2b + 3tc + d) = f(1, 0)\prod_{t=0}^{p-1} f(t, 1),$$

whose vanishing is the condition that one of the points (x, y) represented by $f = 0$ shall be one of the existing $p + 1$ real points $(1, 0)$, $(t, 1)$ of the modular line. To verify algebraically that the seminvariant (48) is an invariant,* note that it is unaltered modulo p by the substitution

$$(49) \qquad a' = -d, \quad d' = a, \quad b' = c, \quad c' = -b,$$

which is induced on the coefficients of f by $x = y'$, $y = -x'$.

The product P of the δ_{jk} in which j and k are such that

$$\lambda = t^3 - 3kt - j$$

is irreducible modulo p is a formal invariant.

The substitution (49) replaces the general factor of (47) by

$$-a + 3tb - 3(t^2 - k)c + \lambda d$$
$$= \lambda\{(T^3 - 3KT - J)a + 3(T^2 - K)b + 3Tc + d\},$$

where

$$T = \frac{k - t^2}{\lambda}, \qquad K = \frac{g}{\lambda^2}, \qquad J = \frac{h}{\lambda^3}, \qquad g = k^2 + kt^2 + tj,$$

$$h = -2k^3 + 6k^2t^2 + 3ktj + t^3j + j^2.$$

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* For any form, see *Transactions of the American Mathematical Society*, vol. 8 (1907), pp. 207–208.

First, let $\beta_i = \beta$. Then $i \equiv j$, $e_i \equiv 0 \pmod{n}$, whence $i = j$. Thus the exponent of a in any term (42) or (43) is divisible by n, while the exponent of c is not, being congruent to j modulo n. Hence the coefficient of b^β in the sum of (42) and the various terms (43), with $i = j$, is not symmetrical in a and c, unless identically zero. But (43) has the factor a while (42) does not. Hence the greatest β_i exceeds β.

Next, consider a set of terms (43) and a set of terms of like form with i replaced by k, all being of equal degree in b. Then $\beta_i = \beta_k$. By (44_1), $2i + \beta_i \equiv 2k + \beta_k$, $i = k$. Consider finally terms (43) with β_i constant. In them the residue modulo n of e_i is a constant $\neq i$. For, if $e_i \equiv i$, then $2i + \beta_i \equiv j + \beta \pmod{n}$ by (44_2), so that $j \equiv 0 \pmod{n}$ by (44_1). Hence these terms (43) are not symmetric in a and c and yet do not cancel.*

Our fundamental invariants are connected by a syzygy; for $p = 3$,

$$B^2 = \Delta^3\Gamma^2 + J(J - \Delta^2)^2. \tag{45}$$

9. *Formal Invariants of a Binary Cubic Form for* $p \neq 3$.—We have seen that the theory of formal invariants of a binary quadratic form is dominated by the invariantive products of linear functions of the coefficients. While these products depended upon the classification of integers into the quadratic residues and the non-residues of p, we shall find that for a cubic form it is a question not merely of cubic residues and non-residues of p, but of the larger classes of reducible and irreducible congruences. Write

$$f = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

thus taking $p \neq 3$. Under transformation (21), f becomes f', whose coefficients are given by (26) and

$$d' = a + 3b + 3c + d. \tag{46}$$

* If two are of like degree in c, their g's are equal and hence their f's are equal; then, if of like degree in a, their e's are equal. But then we have the same term of ϕ_i.

and an absolute formal invariant* K of degree $p-1$. For $p=5$,

(54) $K = b^4 + c^4 - b^2d^2 - a^2c^2 - bc^2d - ab^2c + acd^2 + a^2bd.$

Thus, for $p=5$, K and the discriminant D are invariants of degree 4, and weights $\equiv 0, 2 \pmod 4$, while $a\delta_{00}$ and G are of degree 6 and weight $\equiv 3 \pmod 4$. It follows from § 10 that there are no further invariants of degree less than 8. Now the first and second invariants (51) are of degree 10 and weight $\equiv 1 \pmod 4$. Hence if either is expressible as a polynomial in invariants of lower degrees, it must be the product of D by a linear function of $a\delta_{00}$ and G. This is seen to be impossible either by a consideration of the terms of degree $\geqq 5$ in d or by noting that D has no linear factor. Thus $\gamma_1\delta_{03}$ or $\gamma_4\delta_{02}$ occurs in a fundamental system of invariants.

Invariantive products of linear functions of the coefficients of the cubic form therefore play an important rôle in the theory of its formal invariants. Whether or not they play as dominant a rôle as in the case of the quadratic form is not discussed here. We shall however treat more completely the seminvariants.

10. *Formal Seminvariants of a Binary Cubic for* $p>3$.—We shall first determine the character of the function to which any seminvariant $S(a, b, c, d)$ reduces when $a=0$. Set $A=3b$, $2B=3c$, $C=d$. Then (26) and (46) give

$$A' = A, \quad B' = A + B, \quad C' = A + 2B + C \qquad (\text{when } a = 0).$$

Any function unaltered by this transformation is (§ 7) a polynomial in A, $B^2 - AC$, γ_0', or the product of such a polynomial by β', where γ_0' and β' are the functions γ_0 and β written in capitals. But

$$\gamma_0' = \prod_{t=0}^{p-1} (3t^2b + 3tc + d) = [\delta_{j0}]_{a=0},$$

$$\beta' = \prod_{t=0}^{p-1} \{\tfrac{3}{2}(2tb + c)\} \equiv [\gamma_k]_{a=0},$$

* *Transactions of the American Mathematical Society*, vol. 8 (1907), p. 221; vol. 10 (1909), p. 154, foot-note. *Bulletin of the American Mathematical Society*, vol. 14 (1908), p. 316. Cf. Hurwitz, *l. c.*

We are to show that there is no integral solution x of

$$x^3 - 3Kx - J \equiv 0 \qquad (\mathrm{mod}\ p).$$

Multiply this by λ^3 and set $\lambda x = y$. Then

$$y^3 - 3gy - h \equiv 0 \qquad (\mathrm{mod}\ p).$$

But the negative of the left member is the result of substituting

$$r + s = -t, \quad rs = -y - 2k$$

in the expansion of the product

$$(r^3 - 3kr - j)(s^3 - 3ks - j).$$

The latter is congruent to zero modulo p for no values of r and s which are integers or the roots of an irreducible quadratic congruence with the integral coefficients t, $-y-2k$.

For $p = 2$, $P = \delta_{11}$. For $p = 5$, P is the product of two invariants*

$$\delta_{11}\delta_{22}\delta_{32}\delta_{41}, \quad \delta_{13}\delta_{24}\delta_{34}\delta_{43}, \tag{50}$$

neither of which is a product of invariants. The last property is true also of the following invariants:

$$\begin{gathered}\gamma_1\delta_{03}, \quad \gamma_4\delta_{02}, \quad \gamma_2\delta_{04}\delta_{12}\delta_{30}\delta_{20}\delta_{42}, \\ \gamma_3\delta_{01}\delta_{10}\delta_{23}\delta_{33}\delta_{40}, \quad \beta\gamma_0\delta_{14}\delta_{21}\delta_{31}\delta_{44}.\end{gathered} \tag{51}$$

The product of these seven invariants and $a\delta_{00}$ equals the product of all the linear functions of a, b, c, d, not proportional modulo 5.

For $p = 2$, each of the 15 linear functions is a factor of just one of the following invariants (no one with an invariant factor):

$$a\delta_{00}, \quad \delta_{11}, \quad \beta\gamma_0\delta_{01}, \quad K = b + c, \quad (a + b + c)\delta_{10}. \tag{52}$$

For any $p \neq 3$, the cubic form has the formal invariant

$$G = 3(bc^p - b^pc) - (ad^p - a^pd), \tag{53}$$

* In those linear factors of the first which lack c, the product of the coefficients of a and b is a quadratic non-residue of 5; in those of the second invariant, a quadratic residue.

$a\delta_{00}^g$ with the term ad^{5g}, while d does not occur to this power in the increment to a function λ of degree $5g$. Again, the increment to $q\delta_{00}^h$ has the term $2ad^{1+5h}$, while the increment to a function λ of degree $5h+1$ is of smaller degree in d. Hence $\rho = \sigma = 0$. Then in $a\lambda$, λ is a seminvariant which may be treated as was the initial S.

A fundamental system of formal seminvariants of the binary cubic form modulo 5 *is given by the functions* (59).

11. For $p = 2$, the method of § 10 fails. In place of c we now introduce the seminvariant $K = b + c$. Then the transformation (26), (46), becomes

$$(60) \quad a' = a, \quad K' = K, \quad b' = a + b, \quad d' = a + K + d.$$

By § 3, any seminvariant $S(a, K, b, d)$ becomes for $a = 0$ a polynomial in K, b, $d(K + d)$. In place of the last we may use δ_{00}. Hence

$$S = a\sigma + \phi(b, K, \delta_{00}), \quad \delta_{00} = d(a + K + d).$$

We make use of the seminvariants

$$(61) \quad \begin{gathered} \Delta = ad + bc = \delta_{00} + \delta_{01}, \quad \beta = b^2 + ab, \\ \beta + \Delta = bK + a(b + d). \end{gathered}$$

Hence S differs from a polynomial in K, δ_{00}, Δ, β by a function $a\rho + b\tau(\beta, \delta_{00})$. Let (60) replace ρ by ρ'. Then $\rho + \rho' \equiv \tau \pmod 2$. Take $a = K = 0$; then (60) is the identity and $0 \equiv \tau(b^2, d^2)$ identically in b, d. Hence the function $\tau(\beta, \delta)$ is identically zero. Thus $a\rho$ and hence ρ is a seminvariant. Hence a, K, δ_{00}, Δ, β *form a fundamental system of formal seminvariants of the cubic modulo* 2.

Note that Δ^2 is the discriminant, so that Δ is an invariant. The invariants (52) may be expressed in terms of our seminvariants:

$$(62) \quad \begin{gathered} \delta_{11} = I + \Delta, \quad \beta\gamma_0\delta_{01} = \beta(\beta + K^2 + aK)(\Delta + \delta_{00}), \\ (a + K)\delta_{10} = (a + K)(a^2 + I) = a\delta_{00} + KI, \end{gathered}$$

where $I = a^2 + aK + \delta_{00}$ is an invariant.

modulo p. Hence

$$S = a\sigma(a, b, c, d) + \gamma_k{}^e\phi(b, q, \delta_{j0}) \quad (e = 0 \text{ or } 1), \tag{55}$$

where k, j may be given any assigned integral values and

$$q = c^2 - \tfrac{4}{3}bd, \quad -3b^2q = [D]_{a=0}, \tag{56}$$

D being the discriminant of f. We use the seminvariants (II, § 2)

$$S_2 = -b^2 + ac, \quad S_3 = 2b^3 + a(ad - 3bc). \tag{57}$$

First, let $p = 5$. Then $q \equiv c^2 + 2bd$. We have the formal seminvariants*

$$\begin{aligned}
\sigma_3 &= bq - a(ab + 2cd),\\
\sigma_4 &= K - S_2{}^2 = q^2 + a(abd - 2ac^2 + b^2c + cd^2),\\
\sigma_5 &= bq^2 + a(-ad^3 - bcd^2 + 3c^3d + abc^2 - 2b^3c + a^3b),\\
\sigma_6 &= q^3 + a(ad^4 - 2bcd^3 - c^3d^2 + abc^2d - 2b^3cd + a^3bd + 2ac^4\\
&\qquad - b^2c^3 - 2a^3c^2 + ab^4),\\
\sigma_7 &= q\gamma_0 + a\{2(b^2 - ac)d^4 + a^2bd^3 - bc^2d^3 - 2c^4d^2 + 2a^2c^2d^2\\
&\qquad - 2ac(b^2 - ac)d^2 - (b^2 - ac)^2d^2 - 2a^4d^2 + 2abc^3d\\
&\qquad + 2a^3bcd + 2ab^4c + 3(b^2 - ac)c^4 - a^4b^2 + 2a^3c^3\},
\end{aligned} \tag{58}$$

while $2G$ differs from $b\gamma_0$ by a multiple of a. By (55)–(58), S differs from a polynomial in the seminvariants

$$a,\ D,\ S_2,\ S_3,\ \sigma_3,\ K,\ \sigma_5,\ \sigma_6,\ \sigma_7,\ G,\ \gamma_0,\ \delta_{00} \tag{59}$$

by a function $a\lambda + \rho b\delta_{00}^g + \sigma q\delta_{00}^h$, in which ρ and σ are constants at least one of which is zero (in view of the degree of the terms). But the increment to $b\delta_{00}^g$ under transformation (26), (46), is

* As the terms with the factor a were taken all of the proper degree and weight; then a term common to a combination of the seminvariants (59) was deleted. Finally the coefficients were found by a process equivalent to the use of a (non-linear) annihilator, *Transactions of the American Mathematical Society*, vol. 8 (1907), p. 205. Expansions were made in powers of d and the terms involving d rechecked. As each remaining term involves a new coefficient, there is no doubt as to the existence of covariants of type σ_5, σ_6, σ_7, though the terms free of d were not rechecked.

to $a_0^s a_1^s$. Hence $C \equiv P_1 a_2$. Similarly,

$$L_4/L_3 \equiv -a_3^{p^3} + a_3^{p^2} Q_{32} - a_3^p Q_{31} + a_3 L_3^{p-1} \qquad (\text{mod } p),$$

where the Q's are defined by (16) and are congruent to*

$$Q_{31} = Q(L_3/L_2)^{p-1} + L_2^{p^2-p}, \quad Q_{32} = (L_3/L_2)^{p-1} + Q^p,$$

with Q as above. Hence for integral values of the a's,

$$Q_{31} \equiv (1 - P_1)P_1 a_2^{p-1} \equiv 0, \quad Q_{32} \equiv 1 - P_1(1 - a_2^{p-1}) = 1 - P_2,$$

$$L_4/L_3 \equiv -P_2 a_3.$$

13. *Modular Covariants.*—Extending the usual definition of a covariant of an algebraic form f to the case in which the group is the set of all linear transformations with integral coefficients taken modulo p, we obtain the concepts modular covariants or formal modular covariants according as the coefficients of f are integers taken modulo p or are indeterminates. The contrast is the same as in § 5. The universal covariants obtained in § 2 and § 4 do not involve the coefficients of f and hence are formal covariants.

I have recently proved† that *all rational integral modular covariants of any system of modular forms are rational integral functions of a finite number of these covariants.* In the same paper I proved that *a fundamental system of modular covariants of the binary quadratic form* (25) *modulo* 3 *is given by the form f itself, its discriminant Δ, the universal covariants L and Q, together with*‡

$$q = (a + c)(b^2 + ac - 1), \quad f_4 = ax^4 + bx^3y + bxy^3 + cy^4,$$

$$(63) \quad C_1 = (a^2b - b^3)x^2 + 2(b^2 + ac)(c - a)xy + (b^3 - bc^2)y^2,$$

$$C_2 = (\Delta + a^2)x^2 - 2b(a + c)xy + (\Delta + c^2)y^2.$$

Here f_4 is a formal covariant, which is congruent to f for integral

* *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 77.

† *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 299–310. The extension to cogredient sets of variables has since been made by Professor F. B. Wiley, and will be published in his Chicago dissertation.

‡ No one of the eight is a rational integral function of the remaining seven even in the case of integral coefficients a, b, c taken modulo 3.

12. *Miss Sanderson's Theorem.**—Given a modular invariant i of a system of forms under any modular group G, we can construct a formal modular invariant I of the system of forms under G such that $I \equiv i \pmod{p}$ for all integral values of the coefficients of the forms. As the proof does not give a simple method of actually constructing I from i, it is in place here to give a very interesting illustration of the theorem with independent verification. Take as i the fundamental seminvariant $(-1)^m P_{m-1} a_m$ of a binary form f (Lecture II). Then I is the quotient L_{m+1}/L_m, where L_m is given by (16) or (17) with $x_1, \cdots, x_m$ replaced by the first m coefficients $a_0, a_1, \cdots, a_{m-1}$ of the binary form f. Now $x = x' + y'$, $y = y'$, replaces $f(x, y)$ by a form in which the coefficient a_j' is a linear function of $a_0, \cdots, a_j$. Hence L_j is a formal seminvariant of f modulo p. First,

$$\frac{L_2}{L_1} = \begin{vmatrix} a_0^p & a_1^p \\ a_0 & a_1 \end{vmatrix} \div a_0 = a_0^{p-1} a_1 - a_1^p$$

is a formal seminvariant which reduces to $-P_0 a_1$ for integral values of a_0, a_1, where $P_0 = 1 - a_0^{p-1}$. Compare (27). Next,

$$L_3 = \begin{vmatrix} a_0^{p^2} & a_1^{p^2} & a_2^{p^2} \\ a_0^p & a_1^p & a_2^p \\ a_0 & a_1 & a_2 \end{vmatrix},$$

$$C = L_3/L_2 \equiv a_2^{p^2} - a_2^p Q + a_2 L_2^{p-1} \pmod{p},$$

where, as in (10),

$$Q = \frac{a_0^{p^2} a_1 - a_0 a_1^{p^2}}{L_2} = \sum_{j=0}^{p} a_0^{s(p-j)} a_1^{sj} \quad (s = p - 1).$$

For integral values of the a's, we have

$$L_2 \equiv 0, \quad Q \equiv a_0^s + a_1^s + (p-1) a_0^s a_1^s \equiv 1 - P_1,$$

$$P_1 = (1 - a_0^s)(1 - a_1^s),$$

modulo p, since each term of Q, with $j \neq 0$, $j \neq p$, is congruent

* *Transactions of the American Mathematical Society*, vol. 14 (1913), p. 490.

First, let K be of even order $2n$. Then

$$K_1 = K - IQ^n - rf^n - sbf^n$$

is a covariant in which the coefficient of x^{2n} is zero and hence has the factor y. Thus K_1 has the factor L and the quotient is a covariant of order $2n - 3$ to which the next argument applies.

Next, let K be of odd order:

$$K = Sx^{2n+1} + S_1x^{2n}y + \cdots.$$

After subtracting from K constant multiples of lQ^n and blQ^n, in which the coefficients of x^{2n+1} are $a + b$ and $ab + b$, respectively, we may assume that S is an invariant. After also subtracting from K a constant multiple of ILQ^{n-1}, where I is a linear combination of the invariants (24′) and unity, we may assume that $S_1 = \beta_1 a + \beta_2 c$, where the β's are functions of b only. Then the covariance of K with respect to the transformation (21) gives

$$Sx'^{2n+1} + S_1'x'^{2n}y' + \cdots \equiv K \equiv Sx'^{2n+1} + (S + S_1)x'^{2n}y' + \cdots \pmod 2,$$

where S_1' denotes the function S_1 formed for the new coefficients (22). Hence

$$S_1' - S_1 = \beta_2(a + b)$$

must equal the invariant S. Since $\beta_2 b$ is a function of the invariant b, $\beta_2 a$ must be an invariant, so that $\beta_2 = 0$. Thus $S = 0$ and K has the factor L as before. Hence the theorem is true for covariants of order ω if true for those of order $\omega - 3$. But it was proved true for those of order zero.

By a similar method I obtain the following theorem:

A fundamental system of covariants of the binary quadratic form f, given by (20), *and the linear form $\lambda = a_2x + a_1y$ modulo* 2 *is given by f, λ, l,*

$$l_1 = (aa_2 + j)x + (ca_1 + j)y,$$

Q, L and the invariants b, q, $(a_1 - 1)(a_2 - 1)$ and

$$j = (a + b)a_1 + (b + c)a_2.$$

values of x, y. Also C_2 and (as here written) C_1 are formal covariants. Note that $-q$ is the invariant (42) of Lecture II. When q is made homogeneous by replacing $-a-c$ by $-a^3-c^3$, we obtain the formal invariant $\Gamma=\gamma_2$, given by (32). The resulting eight formal covariants of f do not form a fundamental system of formal covariants; not all the formal invariants are polynomials in Δ and Γ (§ 8). No instance of a fundamental system of formal covariants has yet been published.

The method of proof will be here illustrated by the new and simpler case of a binary quadratic form (20) with integral coefficients modulo 2. By § 6 any invariant of f is a polynomial in

$$(24') \qquad b, \quad abc, \quad q=(b+1)(a+c)+ac,$$

to which the formal invariants (24) reduce modulo 2. Such a polynomial is congruent to a linear function of these three and unity, since

$$bq\equiv abc \pmod{2}.$$

Further, any seminvariant is a polynomial in a, b and q (§ 6), and hence is a linear function of $1, a, b, ab, q, abc$. For,

$$aq\equiv a+ab+abc \pmod{2}.$$

These results are in accord with those obtained otherwise in § 14 of Lecture II. We shall now prove the following theorem:

Every rational integral covariant K of the binary quadratic form f modulo 2 is a rational integral function of f, its invariants b and q, the universal covariants

$$Q=x^2+xy+y^2, \quad L=x^2y+xy^2,$$

and the linear covariant

$$l=(a+b)x+(b+c)y, \quad l^2\equiv f+bQ \pmod{2}.$$

The leading coefficient S of K is a seminvariant and hence is of the form $I+ra+sab$, where r and s are constants, and I is an invariant, a linear combination of the invariants (24′) and unity.

First, let $\lambda \not\equiv 0$. For $z \equiv x$ or $z \equiv y$, we have

$$0 \equiv \begin{vmatrix} x^{p^2} & y^{p^2} & z^{p^2} \\ x^p & y^p & z^p \\ x & y & z \end{vmatrix} \equiv Lz^{p^2} - QLz^p + L^p z \pmod{p}.$$

Hence x and y are roots of

(66) $$F(z) = z^{p^2} - \mu z^p + \lambda z \equiv 0 \pmod{p}.$$

Having no double root, this congruence has p^2 distinct integral or imaginary roots. These roots are

(67) $$eX + fY \quad (e, f = 0, 1, \cdots, p - 1),$$

where X and Y are particular roots linearly independent modulo p. For,

(68) $$F(eX + fY) = eF(X) + fF(Y).$$

Hence any pair of solutions x, y of (65) is of the form (7), where $a, \cdots, d$ are integers, whose determinant Δ is not divisible by p, in view of (64_1) and $\lambda \not\equiv 0$.

Conversely, if X and Y are fixed linearly independent solutions of (66), any pair of linear functions of X and Y with integral coefficients, whose determinant is not divisible by p, gives a solution of (65). Indeed, by (68), x and y are solutions of (66). From the two resulting identities, we eliminate λ and μ in turn and get

$$\mu = Q(x, y), \quad \{L(x, y)\}^p = \lambda L(x, y).$$

Since X and Y are linearly independent modulo p, $L(X, Y)$ is not divisible by p [cf. (6)]. Thus $L(x, y) \not\equiv 0$ by (64). Hence (65) hold.

Hence, for $\lambda \not\equiv 0$, the form problem has been reduced to the solution of congruence (66). The latter will be discussed here in the simple but typical* case in which λ and μ are integers. Now the problem to find the real and imaginary roots of a con-

* For the general case, see *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 87.

Since a_1 and a_2 are cogredient with x and y, the function j obtained from the covariant l of f is an invariant of f and λ.

The reverse of the last process is important. If we adjoin to a system of binary forms in the variables x' and y' the linear form $yx' - xy'$, any modular invariant of the enlarged system, formal as to x, y, is a modular covariant of the given system with x', y' replaced by x, y. The theorem of § 12 therefore proves the existence of certain formal covariants.*

Applications of Invariants of a Modular Group, §§ 14, 15

14. *Form Problem for the Total Binary Modular Group* Γ.— This group is composed of all binary linear transformations (7) with integral coefficients taken modulo p whose determinant Δ is not divisible by p. By (8),

$$(64)\quad L(x, y) \equiv \Delta L(X, Y), \quad Q(x, y) \equiv Q(X, Y) \pmod{p},$$

so that L^{p-1} and Q are absolute invariants of Γ. Hence, of the functions (11), q is invariant under Γ, while l is unaltered by certain transformations and changed in sign by others. Thus a homogeneous function of q and l having a term which is a power of q is a relative invariant of Γ only when an absolute invariant. Hence if $p > 2$, it involves only even powers of l, and by the homogeneity, only even powers of q. Hence *any absolute invariant of* Γ *is a product of powers of* L^{p-1} *and* Q *by a polynomial in* q^γ, l^γ, *where* $\gamma = 1$ *if* $p = 2$, $\gamma = 2$ *if* $p > 2$.

In particular, L^{p-1} and Q form a fundamental system of absolute invariants of Γ. The so-called form problem for the group Γ requires the determination of all pairs of values of the variables x and y for which L^{p-1} and Q are congruent modulo p to assigned values λ and μ, either integers or imaginary roots of congruences modulo p. We have therefore to solve the system of congruences

$$(65)\qquad \{L(x, y)\}^{p-1} \equiv \lambda, \quad Q(x, y) \equiv \mu \pmod{p}.$$

* After these lectures were delivered, I saw a manuscript by Professor O. E. Glenn, containing tables of formal concomitants for forms of low orders and moduli 2 and 3. He employs transvection between the form and the covariant L of § 2.

But $l_D = z$ implies that $l_{D+1} = z^p$. The condition for the latter is therefore $S^D = 1$. Hence D is the period of S. But (69) is the characteristic determinant of S. According as it has distinct roots v_1 and v_2 or equal roots $v = \frac{1}{2}\mu = \lambda^{\frac{1}{2}}$, a linear substitution of matrix S can be transformed linearly into one of matrix*

$$\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad \begin{pmatrix} v & v \\ 0 & v \end{pmatrix}.$$

According as the characteristic congruence (69) *has distinct* (*real or imaginary*) *roots or a double root, D is the least common multiple of the exponents to which the distinct roots belong modulo p, or is p times the exponent to which the double root belongs.*

Finally, let $\lambda \equiv 0$. By (6), either $y \equiv 0$ or $x - ay \equiv 0 \pmod{p}$, where a is an integer. In the first case,

$$Q = x^{p^2-p}, \quad x^{p^2} - \mu x^p = 0.$$

If $\mu = 0$, then $x = y = 0$. If $\mu \neq 0$, the roots x are equal in sets of p and hence are cx_1 ($c = 0, 1, \cdots, p-1$), where x_1 is a particular root not divisible by p. In the second case $x - ay \equiv 0$, we take $x - ay$ as a new variable X and conclude from the absolute invariance of Q that

$$Q(x, y) = Q(0, y) = y^{p^2-p}.$$

We thus have the first case with y in place of x.

Using similar methods, I have solved the form problem for the total group of modular linear transformations on m variables.†

15. *Invariantive Classification of Forms.*—Let

$$\phi(x, y) = x^m + \cdots \qquad (m > 1) \tag{70}$$

be a binary form irreducible modulo p and having unity as the coefficient of the highest power of x. Let G be the group of all modular binary linear transformations (1) with integral coef-

* In the second case we use the new variables x and $x - vy$.

† *Transactions of the American Mathematical Society*, vol. 12 (1911), pp. 84–92.

gruence with integral coefficients is at bottom the problem to factor it into irreducible congruences with integral coefficients.

When v is an integer, $z^p - vz$ is a factor of (66) if and only if v is a root of the characteristic* congruence

$$v^2 - \mu v + \lambda \equiv 0 \pmod{p}. \tag{69}$$

Such a binomial is a product† of binomials $z^d - \delta$, irreducible modulo p, whose degree d is the exponent to which the integer v belongs modulo p. Since $2p - 1 < p^2$, the function (66) has an irreducible factor $\phi(z)$ of degree $D > 1$, not of the preceding type $z^d - \delta$, and hence with a root r such that r^p/r is not congruent to an integer. Thus every root of (66) is of the form $c_1r + c_2r^p$, where the c's are integers. *The irreducible factors of* (66) *are of degree D except those, occurring only when* (69) *has an integral root, of the form $z^d - \delta$, where d is a divisor of D.*

To find D, note that by raising (66) to the powers $p, p^2, \cdots$, we can express z^{p^t} as a linear function l_t of z^p and z. Now D is the least value of t for which $l_t \equiv z$. But the coefficients of l_t are the elements of the first row of the matrix of S^{t-1}, where

$$S = \begin{pmatrix} \mu & -\lambda \\ 1 & 0 \end{pmatrix}.$$

* Note the analogy of (66) with the linear differential equation

$$F(z) = \frac{d^2z}{dt^2} - \mu \frac{dz}{dt} + \lambda z = 0,$$

having the solution $z = e^{vt}$ if v is a root of $v^2 - \mu v + \lambda = 0$. Also, (68) holds. Make dz/dt correspond to z^p and hence d^2z/dt^2 to $(z^p)^p$. Thus the differential equation corresponds to (66), and the integral $z = e^{vt}$ (viz., $dz/dt = vz$) to $z^p = vz$.

† Let $f(z)$ be an irreducible factor of degree d. Its roots are

$$r, \quad r^p \equiv vr, \quad r^{p^2} \equiv v^2r, \quad \cdots, \quad r^{p^{d-1}} \equiv v^{d-1}r,$$

where $v^d \equiv 1$, $v^l \not\equiv 1$, $0 < l < d$. Thus d is a divisor of $p - 1$. Hence

$$z^{p-1} - v = z^{p-1} - r^{p-1}$$

has the factor $z^d - r^d$. The latter has a root r in common with $f(z)$. But

$$(r^d)^{p-1} \equiv v^d \equiv 1.$$

Thus $\delta = r^d$ is an integer. Hence $f(z) = z^d - \delta$.

In general, let m be a product of powers of the distinct prime numbers $q_1, \cdots, q_\mu$, and set

$$F_t = (x^{p^t}y - xy^{p^t})/L.$$

From the expression for π_m due to Galois, we readily obtain

$$\pi_m = \frac{F_m \cdot \Pi F_{m/q_iq_j} \cdot \Pi F_{m/q_iq_jq_kq_l} \cdots}{\Pi F_{m/q_i} \cdot \Pi F_{m/q_iq_jq_k} \cdots},$$

in which the first product in the numerator extends over the $\frac{1}{2}\mu(\mu - 1)$ combinations of $q_1, \cdots, q_\mu$ two at a time, and similarly for the remaining products. By the first theorem of this section, and (11), π_m is a polynomial in

$$J = q^\gamma = Q^{p+1}, \quad K = l^\gamma = L^{p(p-1)} \quad (\gamma = 1 \text{ if } p=2, \gamma=2 \text{ if } p>2).$$

We readily verify the recursion formula

$$F_t \equiv QF_{t-1}^p - KF_{t-2}^{p^2} \qquad (\text{mod } p),$$

since $F_1 = 1$, $F_2 = Q$. In particular,

$$F_3 \equiv J - K, \quad F_4 \equiv Q(F_3{}^p - KJ^{p-1}).$$

Now $\pi_3 = F_3$, $\pi_4 = F_4/Q$. Hence

$$\pi_3 \equiv J - K, \quad \pi_4 \equiv J^p - K^p - KJ^{p-1} \qquad (\text{mod } p).$$

The first of these results was discussed above. Next, for $p = 2$, π_4 is the irreducible quadratic form $q^2 - l^2 - lq$, so that all quartic forms irreducible modulo 2 are equivalent. For $p > 2$, π_4 vanishes for $K = \rho J$, where

$$\rho^p \equiv 1 - \rho \qquad (\text{mod } p).$$

Except for $\rho \equiv \frac{1}{2}$, ρ is a quadratic Galois imaginary since

$$\rho^{p^2} \equiv 1 - \rho^p \equiv \rho \qquad (\text{mod } p).$$

Thus π_4 is a product of $J - 2K$ and $\frac{1}{2}(p - 1)$ irreducible quadratic forms in J, K. Some of the latter yield a quartic in q and l which is irreducible; others yield a quartic which is a product of two irreducible quadratics modulo p. A simple discussion shows

ficients of determinant unity. Let $\phi_1 = \phi, \phi_2, \cdots, \phi_k$ denote all the forms of type (70) which can be transformed into constant multiples of ϕ by transformations of G. Evidently their product $P = \phi_1\phi_2 \cdots \phi_k$ is transformed into c_tP by any transformation t of G. The constant c_t is easily seen* to be congruent to unity. Hence P is an absolute invariant of G. If $m > 2$, no ϕ_i vanishes for a special point. We now apply the theorem in the first part of § 14. Hence, *if $m > 2$, the absolute invariant P is an integral function with integral coefficients of the invariants q, l, each exponent of q and l being even if $p > 2$.* In view of the definition of the ϕ_i, this function of q and l is an irreducible function of those arguments modulo p.

Two binary forms shall be said to be *equivalent* if and only if one of them can be transformed into a constant multiple of the other by a transformation of G. A set of all forms equivalent to a given one shall be called a *genus*. Thus $\phi_1, \cdots, \phi_k$ form a genus. All of the irreducible forms (70) separate into a finite number f of distinct genera; let $P_1, \cdots, P_f$ denote the products of the forms in the respective genera. Thus $\pi_m = P_1 \cdots P_f$ is the product of all of the binary forms $x^m + \cdots$ irreducible modulo p. Hence π_m is a polynomial in q, l with integral coefficients. Hence *the f genera of irreducible binary forms of degree $m > 2$ are characterized invariantively by the f irreducible factors $P_i(q, l)$ of $\pi_m(q, l)$ modulo p.*

We shall see that $\pi_m(q, l)$ is easily computed. By finding its factors irreducible modulo p in the arguments q, l, we shall have invariantive criteria for the equivalence of two irreducible binary forms of degree m. For example, we shall prove that $\pi_3 = q - l$ if $p = 2$, so that all irreducible binary cubic forms modulo 2 are equivalent. Further, $\pi_3 = q^2 - l^2$ if $p > 2$, so that the irreducible cubic factors of $q - l$ are all equivalent, also those of $q + l$, while no factor of the former is equivalent to one of the latter.

* *Transactions of the American Mathematical Society*, vol. 12 (1911), p. 3, § 4. The present section is an account of the simpler topics there treated at length.

LECTURE IV

MODULAR GEOMETRY AND COVARIANTIVE THEORY OF A QUADRATIC FORM IN m VARIABLES MODULO 2

1. *Introduction.*—The modular form that has been most used in geometry and the theory of functions is the quadratic form

$$(1) \qquad q_m(x) = \Sigma c_{ij}x_ix_j + \Sigma b_ix_i^2 \quad (i, j = 1, \cdots, m;\ i < j)$$

with integral coefficients taken modulo 2. In accord with Lecture III, we shall use the term point to denote a set of m ordered elements, not all zero, of the infinite field F_2 composed of the roots of all congruences modulo 2 with integral coefficients. We shall identify such a point $(x_1, \cdots, x_m)$ with $(\rho x_1, \cdots, \rho x_m)$ where ρ is any element not zero in F_2. The point is called real if the ratios of the x's are congruent to integers modulo 2. Let the c_{ij} and b_i in (1) be elements not all zero of the field F_2. Then the aggregate of the points $(x) = (x_1, \cdots, x_m)$ for which $q_m(x) \equiv 0 \pmod 2$ shall be called a quadric locus, in particular, a conic if $m = 3$. The locus is thus composed of an infinitude of points, a finite number of which are real.

While our results are purely arithmetical, we shall find that the employment of the terminology and methods of analytic projective geometry is of great help in the investigation. Usually the proofs are given initially in an essentially arithmetical form. In case a preliminary argument is based upon geometrical intuition, a purely algebraic proof is given later. The geometry brings out naturally the existence of a linear covariant, which is important in the problem of the determination of a fundamental system of covariants.

2. *The Polar Locus.*—The point $(\kappa y_1 + \lambda z_1, \cdots, \kappa y_m + \lambda z_m)$ is on $q(x) \equiv 0$ if

$$(2) \qquad \kappa^2 q(y) + \kappa\lambda P(y, z) + \lambda^2 q(z) \equiv 0 \pmod 2,$$

that the number of irreducible factors of $\pi_4(q, l)$ is $6k + t + 1$ if $p = 8k + t$ ($t = \pm 1$ or -3), but is $6k + 2$ if $p = 8k + 3$. We have therefore the number f of genera of irreducible quartics modulo p. For quintics and septics, the analogous discussion is simple, for sextics laborious.

We may utilize similarly the invariants (16) of the group on m variables, obtain expressions in terms of them of the product of all forms in m variables of specified types (as quadratic forms transformable into an irreducible binary form, non-vanishing ternary forms, non-degenerate ternary quadratic forms, etc.), and hence draw conclusions as to the equivalence of forms of the specified type.*

* *Transactions of the American Mathematical Society*, vol. 12 (1911), pp. 92–98.

called the *apex** of the locus $q(x) \equiv 0$. Now each $u_i \equiv 0$ if $z_1 \equiv C_1, \cdots, z_m \equiv C_m$. Hence, *for m odd, the polars of all points pass through the apex.*

If (y) is any point not the apex, the line joining (y) to the apex is tangent to $q(x) \equiv 0$ (§ 2). *Thus any line through the apex is tangent to $q(x) \equiv 0$.*

For $m = 3$, it is true conversely that, if the line

$$\Sigma u_i x_i \equiv 0 \pmod 2 \tag{6}$$

is tangent to $q(x) \equiv 0$, it passes through the apex, so that

$$\kappa = \Sigma C_i u_i \tag{7}$$

is zero modulo 2. Taking, for example, $u_3 \not\equiv 0$, we obtain by eliminating x_3 from (6) and $q(x) \equiv 0$ a quadratic equation in x_1 and x_2 whose left member is the square of a linear function modulo 2 if and only if the coefficient of x_1x_2 is congruent to zero. But this coefficient is the product of κ by a power of u_3. *Thus $\kappa = 0$ is the tangential equation of $q(x) \equiv 0$.*

The last result is true for any odd m. The spread (6) is said to be tangent to $q(x) \equiv 0$ if the locus of their intersections is degenerate. Taking $u_m \not\equiv 0$, and eliminating x_m between (6) and $q(x) \equiv 0$, we obtain a quadratic form whose discriminant, defined by (24), equals a product of κ by a power of u_m, and hence is degenerate if and only if $\kappa \equiv 0$.

We thus have geometrical evidence that *κ is a formal contravariant of $q(x)$*, i. e., an invariant of $q(x)$ and $\Sigma u_i x_i$.

To give an algebraic proof, note that κ is unaltered when x_i and x_j are interchanged, while

$$x_1 = x_1' + x_2', \quad x_2 = x_2', \quad \cdots, \quad x_m = x_m' \tag{8}$$

replaces $q(x)$ by $q'(x')$ in which the altered coefficients are

$$b_2' = b_2 + b_1 + c_{12}, \quad c_{2i}' = c_{2i} + c_{1i} \quad (i = 3, \cdots, m). \tag{9}$$

* After these lectures were delivered, I learned that Professor U. G. Mitchell had obtained, independently of me, the notion apex ("outside point") for the case $m = 3$, Princeton dissertation, 1910, printed privately, 1913.

where

$$P(y, z) = \Sigma c_{ij}(y_i z_j + y_j z_i) \quad (i, j = 1, \cdots, m; i < j). \tag{3}$$

If (y) is a fixed point, all points (z) for which $P(y, z) \equiv 0$ (mod 2) are said to form the polar* locus of (y). For $(z) = (y)$, each summand in (3) is congruent to zero modulo 2. Hence the polar of (y) passes through (y). If (z) is on the polar of (y), (2) has a double root $\kappa : \lambda$ and the line joining (y) and (z) is tangent to $q \equiv 0$.

We may write (3) in the form

$$P(y, z) = u_1 y_1 + \cdots + u_m y_m, \tag{3'}$$

where

$$\begin{aligned} u_1 &= \qquad\qquad\quad c_{12}z_2 + c_{13}z_3 + c_{14}z_4 + \cdots + c_{1m}z_m, \\ u_2 &= c_{12}z_1 \qquad\qquad\;\; + c_{23}z_3 + c_{24}z_4 + \cdots + c_{2m}z_m, \\ u_3 &= c_{13}z_1 + c_{23}z_2 \qquad\qquad\; + c_{34}z_4 + \cdots + c_{3m}z_m, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ u_m &= c_{1m}z_1 + c_{2m}z_2 + c_{3m}z_3 + \cdots + c_{m-1m}z_{m-1}. \end{aligned} \tag{4}$$

There is a striking difference between the cases m odd and m even.

3. *Odd Number of Variables; Apex; Linear Tangential Equation.* Let m be odd. Then the determinant of the coefficients in (4) is congruent modulo 2 to a skew-symmetric determinant of odd order and hence is identically congruent to zero. Hence we can find values of $z_1, \cdots, z_m$ not all congruent to zero such that $u_1, \cdots, u_m$ are all zero modulo 2. Thus *the polars of all points (y) have at least one point in common.*

We shall limit attention to the case in which the pfaffians

$$C_1 = [23 \cdots m], \quad C_2 = [134 \cdots m], \quad \cdots, \quad C_m = [12 \cdots m-1] \tag{5}$$

are not all congruent to zero. The point $(C_1, \cdots, C_m)$ shall be

* Take $\kappa = 1$ and let (z) be a point not on $q(x) \equiv 0$. Then (2) is a quadratic congruence in λ with coefficients in F_2 and hence has two roots λ_1 and λ_2 in that field. Now the points (y) and (z) are separated harmonically by $(y + \lambda_1 z)$ and $(y + \lambda_2 z)$ if and only if $\lambda_1 \equiv -\lambda_2$, that is, if $\lambda_1 \equiv \lambda_2$ (mod 2). But the condition for a double root of (2) is $P \equiv 0$ (mod 2).

of the linear covariant $A_m \Sigma b_i x_i$. We shall see however that there exists a more fundamental linear covariant.

4. *Covariant Line of a Conic.*—Since we shall later treat in detail the case $m = 3$, we shall replace (1) by the simpler notation

$$(14) \quad F(x) = a_1x_2x_3 + a_2x_1x_3 + a_3x_1x_2 + b_1x_1^2 + b_2x_2^2 + b_3x_3^2.$$

Its apex is (a_1, a_2, a_3). Its discriminant (12) is

$$(15) \quad \Delta = F(a_1, a_2, a_3) \equiv a_1a_2a_3 + a_1^2b_1 + a_2^2b_2 + a_3^2b_3.$$

The invariant (13) becomes

$$(16) \qquad A = \alpha_1\alpha_2\alpha_3 \qquad (\alpha_i = a_i + 1).$$

Consider a form (14) with integral coefficients and not the square of a linear function. Then not every a_i is congruent to zero modulo 2. By an interchange of variables we may set $a_3 \equiv 1$. Replace x_1 by $X_1 + a_1x_3$ and x_2 by $X_2 + a_2x_3$. We get

$$X_1X_2 + b_1X_1^2 + b_2X_2^2 + \Delta x_3^2.$$

Let $\Delta \equiv 1$. Replace x_3 by $X_3 + b_1X_1 + b_2X_2$. We get

$$(17) \qquad \phi = X_1X_2 + X_3^2.$$

The only real points on $\phi \equiv 0 \pmod 2$ are $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$. In addition to these and the apex $(0, 0, 1)$, the only real points in the plane are $(1, 1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$. These lie on the straight line

$$(18) \qquad X_1 + X_2 + X_3 \equiv 0 \qquad (\text{mod } 2).$$

Hence with every non-degenerate conic modulo 2 is associated covariantly a straight line.

The inverse of the transformation used above is

$$X_1 = x_1 + a_1x_3, \quad X_2 = x_2 + a_2x_3,$$
$$X_3 = b_1x_1 + b_2x_2 + (1 + a_1b_1 + a_2b_2)x_3.$$

It must therefore replace ϕ by the general form (14) having

The pfaffians $C_2, \cdots, C_m$ are unaltered modulo 2, while

$$(10) \qquad C_1' \equiv C_1 + C_2, \quad u_2' \equiv u_2 + u_1, \quad u_i' \equiv u_i \quad (i \neq 2) \qquad \pmod 2.$$

Hence κ is unaltered modulo 2. Note that

$$(11) \qquad \kappa^2 \equiv \begin{vmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1m} & u_1 \\ c_{12} & 0 & c_{23} & \cdots & c_{2m} & u_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{1m} & c_{2m} & c_{3m} & \cdots & 0 & u_m \\ u_1 & u_2 & u_3 & \cdots & u_m & 0 \end{vmatrix} \pmod 2.$$

We saw that $C_1, \cdots, C_m$ are cogredient with $x_1, \cdots, x_m$. This is evident from the fact that the apex is covariantively related to $q(x)$. Hence if we substitute C_1 for $x_1, \cdots, C_m$ for x_m in (1), we obtain the formal invariant

$$(12) \qquad q_m(C) = \Sigma c_{ij} C_i C_j + \Sigma b_i C_i^2 \quad (i, j = 1, \cdots, m; i < j).$$

If this invariant vanishes, the apex is on the locus, which is then a cone. Indeed, by (2), every point on the line joining (C) to a point on $q(x) = 0$ lies on the latter. Hence $q(x)$ can be transformed into a form in $m - 1$ variables and hence has the discriminant zero. To argue algebraically, let new variables be chosen so that the apex becomes $(0, \cdots, 0, 1)$. The polar of any point (y) passes through the apex. Taking $z_1 = 0, \cdots, z_{m-1} = 0$, $z_m = 1$ in (4), we see that the polar (3′) becomes $c_{1m}y_1 + \cdots + c_{m-1m}y_{m-1}$, which must vanish for arbitrary y's. Hence $b_m x_m^2$ is the only term of (1) involving x_m. But the apex is on the locus. Hence $b_m = 0$ and $q(x)$ is free of x_m. The converse is obvious from (5).

Whether m is odd or even, $q(x)$ has the invariant

$$(13) \qquad A_m = \Pi(c_{ij} + 1) \quad (i, j = 1, \cdots, m; i < j).$$

This is evidently true by (9) or as follows. If $A_m \equiv 1 \pmod 2$, every $c_{ij} \equiv 0$ and $q \equiv (\Sigma b_i x_i)^2$; while if $A_m \equiv 0$, at least one c_{ij} is not congruent to zero, and q is not a double line.

Hence the product $A_m q(x)$ is a covariant; in fact, the square

If $\Delta_m \not\equiv 0 \pmod 2$, we can solve equations (4) for the z's. Substituting the resulting values into $q(z)$, we obtain the tangential equation $U_m \equiv 0$ of $q(x) \equiv 0$. For $m = 2$ and $m = 4$, we get

$$(25)\quad \begin{aligned} U_2 &= c_{12}u_1u_2 + b_2u_1^2 + b_1u_2^2, \\ U_4 &= [1234]\Sigma c_{34}u_1u_2 + \Sigma(c_{23}c_{24}c_{34} + b_2c_{34}^2 + b_3c_{24}^2 + b_4c_{23}^2)u_1^2. \end{aligned}$$

Bordering the algebraic discriminant of (1), we find that

$$(26)\quad 2U_m \equiv \begin{vmatrix} 2b_1 & c_{12} & c_{13} & \cdots & c_{1m} & u_1 \\ c_{12} & 2b_2 & c_{23} & \cdots & c_{2m} & u_2 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\ c_{1m} & c_{2m} & c_{3m} & \cdots & 2b_m & u_m \\ u_1 & u_2 & u_3 & \cdots & u_m & 0 \end{vmatrix} \pmod 4.$$

Finally, let $\Delta_m \equiv 0 \pmod 2$. Then all of the first minors of the matrix of the coefficients in (4) are zero modulo 2. Hence the polars of all points have in common the points of a straight line S. Since its discriminant vanishes, $q(x)$ can be transformed linearly into a quadratic form in $x_1, \cdots, x_{m-1}$, which therefore represents a cone with the vertex $(0, \cdots, 0, 1)$. Let (z) be the vertex of the initial cone $q(x) \equiv 0$. If (x) is any point on the cone, $(x + \lambda z)$ is on the cone, and, by (2), $P(x, z)$ is congruent to zero identically in $x_1, \cdots, x_m$. Hence the linear functions (4) all vanish. Thus the line S meets the cone in its vertex, and z_m^2 is the discriminant of $q_{m-1}(x)$, while z_i^2 is obtained from that discriminant by interchanging m and i. For example, if $m=4$,

$$z_4^2 = c_{12}c_{13}c_{23} + b_1c_{23}^2 + b_2c_{13}^2 + b_3c_{12}^2, \cdots,$$
$$z_1^2 = c_{23}c_{24}c_{34} + b_2c_{34}^2 + b_3c_{24}^2 + b_4c_{23}^2.$$

The product of the general form (1) by $\delta = \Delta_m + 1$ is a quadratic form whose discriminant is zero modulo 2 and hence has the vertex $(\delta z_1, \cdots, \delta z_m)$, where z_i^2 has the value just given. Hence $\delta z_1^2, \cdots, \delta z_m^2$ are cogredient with $x_1, \cdots, x_m$.

6. *Covariant Plane of a Degenerate Quadric Surface.*—The product of q_4 by $\delta = [1234] + 1$ is a quaternary form f whose

$a_3 \equiv \Delta \equiv 1$. It actually replaces (18) by

$$(b_1 + 1)x_1 + (b_2 + 1)x_2 + (b_3 + \alpha_1\alpha_2 + 1)x_3,$$

in which we have added $\Delta + 1 \equiv 0$ to the initial coefficient of x_3. Guided by symmetry, we restore terms which become zero for $a_3 = 1$ and get

$$L = \sum_{i=1}^{3} (\beta_i + 1)x_i, \tag{19}$$

$$\beta_1 = b_1 + \alpha_2\alpha_3, \quad \beta_2 = b_2 + \alpha_1\alpha_3, \quad \beta_3 = b_3 + \alpha_1\alpha_2.$$

Making the terms homogeneous we obtain the formal covariant

$$L = B_1x_1 + B_2x_2 + B_3x_3, \tag{20}$$

$$B_1 = b_1^2 + a_2a_3 + a_2^2 + a_3^2, \quad B_2 = b_2^2 + a_1a_3 + a_1^2 + a_3^2, \tag{21}$$
$$B_3 = b_3^2 + a_1a_2 + a_1^2 + a_2^2.$$

Under the substitution $(a_ia_j)(b_ib_j)$ induced upon the coefficients of F by (x_ix_j), we see that B_i and B_j are interchanged. Under (9), viz.,

$$b_2' \equiv b_2 + b_1 + a_3, \quad a_1' \equiv a_1 + a_2 \pmod 2, \tag{22}$$

there results

$$B_1' \equiv B_1, \quad B_2' \equiv B_2 + B_1, \quad B_3' \equiv B_3 \pmod 2. \tag{23}$$

Hence (20) *is a formal covariant of* F. For other interpretations of L see § 8.

5. *Even Number of Variables.*—The determinant of the coefficients in (4) is congruent modulo 2 to the square of the pfaffian

$$\Delta_m = [123 \cdots m]. \tag{24}$$

This is in fact the discriminant of q_m, which is degenerate if and only if $\Delta_m \equiv 0 \pmod 2$. I have elsewhere* discussed at length the invariants of q_m.

* *Transactions of the American Mathematical Society*, vol. 8 (1907), p. 213 (case $m = 2$); vol. 10 (1909), pp. 133–149; *American Journal of Mathematics*, vol. 30 (1908), p. 263; *Proceedings of the London Mathematical Society*, (2), vol. 5 (1907), p. 301.

These lie by threes in exactly 20 straight lines, which occur in the columns of the table, with the heading "Sides." With these lines we can form exactly 15 complete quadrilaterals, the three diagonals of each of which intersect* in a point on $F \equiv 0$, given in the last column. The columns, with the heading "Plane," give the equations defining the plane of the quadrilateral. In each case, the two equations of the plane have in common with $F \equiv 0$ a single real point, the intersection of the diagonals. Thus the real points on $F \equiv 0$ are its points of contact with these tangent planes.

Sides				Diagonals			Plane	Intersection
146	157	356	347	13	45	67	$x_1=0,\quad x_3+x_4+x_5=0$	01000
146	1*ab*	49*b*	69*a*	19	4*a*	6*b*	$x_2=0,\quad x_3+x_4+x_5=0$	10000
146	1*ef*	4*df*	6*de*	1*d*	4*e*	6*f*	$x_1=x_2=x_3+x_4+x_5$	11001
157	1*ab*	5*ac*	7*bc*	1*c*	5*b*	7*a*	$x_1+x_2+x_3=x_3+x_4+x_5=0$	11011
157	1*ef*	58*e*	78*f*	18	5*f*	7*e*	$x_2=x_3,\quad x_1=x_3+x_4+x_5$	10010
1*ab*	1*ef*	2*ae*	2*bf*	12	*af*	*be*	$x_1=x_3,\quad x_2=x_3+x_4+x_5$	01010
28*c*	29*d*	38*d*	39*c*	23	89	*cd*	$x_3=0,\quad x_1=x_2+x_5$	00010
28*c*	2*ae*	5*ac*	58*e*	25	8*a*	*ce*	$x_4=0,\quad x_1=x_2+x_5$	00100
28*c*	2*bf*	78*f*	7*bc*	27	8*b*	*cf*	$x_3=x_4,\quad x_1=x_2+x_4+x_5$	00111
29*d*	2*ae*	69*a*	6*de*	26	9*e*	*ad*	$x_1=x_3+x_4=x_2+x_5$	01111
29*d*	2*bf*	49*b*	4*df*	24	9*f*	*bd*	$x_1=x_4,\quad x_2=x_3+x_4+x_5$	01100
347	38*d*	4*df*	78*f*	3*f*	48	7*d*	$x_2=x_4,\quad x_1=x_3+x_4+x_5$	10100
347	39*c*	49*b*	7*bc*	3*b*	4*c*	79	$x_4=x_1+x_2=x_3+x_5$	11101
356	38*d*	58*e*	6*de*	3*e*	5*d*	68	$x_2=x_3+x_4,\quad x_1=x_2+x_5$	10111
356	39*c*	5*ac*	69*a*	3*a*	59	6*c*	$x_5=x_1+x_2=x_3+x_4$	11110

8. *Certain Formal and Modular Covariants of a Conic.*—For conic (14), the polar form is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}. \tag{28}$$

Hence if two sets of variables y_i and z_i be transformed cogrediently with the set x_i, this polar form (28) is a covariant of F and the two points (y), (z), in an extended sense of the term

* The dual of the theorem of Veblen and Bussey, "Finite projective geometries," *Transactions of the American Mathematical Society*, vol. 7 (1906), p. 245.

discriminant is zero and hence can be transformed into a form (14) free of x_4. With this cone $F \equiv 0$ is associated covariantively the plane $l = 0$, where l is the ternary covariant (19). Hence f has a linear covariant L which reduces to l when $b_4 \equiv 0$, $c_{i4} \equiv 0$ $(i = 1, 2, 3)$. Relying upon symmetry and the presence of the factor δ, we are led to conjecture that

$$(27)\quad \begin{aligned} L = \delta\{b_1 + 1 + (c_{12} + 1)(c_{13} + 1)(c_{14} + 1)\}x_1 + \cdots \\ + \delta\{b_4 + 1 + (c_{14} + 1)(c_{24} + 1)(c_{34} + 1)\}x_4. \end{aligned}$$

It is readily verified algebraically that L is a covariant of q_4.

There is a simple interpretation of L. If $[1234] \not\equiv 0 \pmod 2$, then $\delta \equiv 0$ and L is identically zero. If $[1234] \equiv 0$, q_4 is degenerate and can be transformed into $\phi = x_1x_2 + x_3^2$ or a form involving only x_1 and x_2. In the former case, $L = x_1 + x_2 + x_3$. Of the 15 real points in space, the seven $(100x)$, $(010x)$, $(111x)$ and (0001) are on the cone $\phi \equiv 0$, the two $(001x)$ are on the invariant line S through the vertex (0001) of the cone and the apex (0010) of the conic cut out by $x_4 \equiv 0$, while the remaining six $(101x)$, $(011x)$, $(110x)$ lie on the plane $L = 0$. Hence with a degenerate quadric surface, not a pair of planes, is associated covariantively a plane, just as a line (19) is associated with a non-degenerate conic (14).

Every linear covariant is of the form IL, where I is an invariant. Every quadratic covariant is a linear combination of the IL^2 and Iq_4.

7. *A Configuration Defined by the Quinary Surface.*—A q_5 whose discriminant is not zero modulo 2 can be transformed into

$$F = x_1x_2 + x_3x_4 + x_5^2.$$

The 15 real points on $F \equiv 0 \pmod 2$ are given in the last column of the table below. In addition to these and the apex (00001) of F, there are just 15 real points in space:

$1=(00011)$, $2=(01001)$, $3=(01011)$, $4=(00101)$, $5=(01101)$,

$6=(00110)$, $7=(01110)$, $8=(10001)$, $9=(10011)$, $a=(10101)$,

$b=(10110)$, $c=(11000)$, $d=(11010)$, $e=(11100)$, $f=(11111)$.

replacement in J and taking t and u_i to be integers, we obtain as the coefficient of $t \equiv t^2$

$$w = \beta_1\beta_2 u_3 + \beta_1\beta_3 u_2 + \beta_2\beta_3 u_1 + \beta_1 u_2 u_3 + \beta_2 u_1 u_3 + \beta_3 u_1 u_2 + u_1 u_2 u_3, \tag{37}$$

a modular invariant of F and λ. By the theorem used above,

$$u = (u_1 + 1)(u_2 + 1)(u_3 + 1) \tag{38}$$

is an invariant of λ. In $w + u + 1$, we replace β_i by the congruent value $B_i + 1$, and render the expression homogeneous in the u's and B's separately. We get

$$\omega = \Sigma(B_1B_2 + B_1^2 + B_2^2)u_3^2 + \Sigma B_1^2 u_2 u_3, \tag{39}$$

a formal invariant of F, λ. For, it is unaltered by the substitution

$$(a_i a_j)(b_i b_j)(u_i u_j),$$

induced by $(x_i x_j)$, and by the substitution (23) and (10) induced by (8). Let the coefficients of F be integers not all even. Then (39) becomes

$$\Sigma(\beta_1\beta_2 + 1)u_3^2 + \Sigma(\beta_1 + 1)u_2 u_3. \tag{39'}$$

Its covariant L is identically zero. Hence, by the table in § 9, if ω is not identically zero it can be transformed into $u_1^2 + u_2^2 + u_1 u_2$ and hence vanishes for a single set of integral values of u_1, u_2, u_3. These are seen to be $u_i = \beta_i + 1$. Hence* *the line* $L = 0$ *is the only line with integral line coordinates on the line locus* (39).

The invariant A for (39) is J (its discriminant is zero, as just seen). Thus a knowledge of any one of the concomitants L, J, ω implies that of the other two.

The covariance of K in (29) implies that

$$\xi_1 = \begin{vmatrix} x_2 & x_3 \\ x_2^2 & x_3^2 \end{vmatrix}, \qquad \xi_2 = \begin{vmatrix} x_1 & x_3 \\ x_1^2 & x_3^2 \end{vmatrix}, \qquad \xi_3 = \begin{vmatrix} x_1 & x_2 \\ x_1^2 & x_2^2 \end{vmatrix} \tag{40}$$

* Also thus: just as the point conic $F \equiv 0$ determines its line equation (36) and hence its apex (a), so the covariant line conic (39) determines the point equation $\Sigma B_i^2 x_i \equiv 0$, which is the line $L \equiv 0$ for integral values of the coefficients.

covariant. In particular, if we take $(y) = (x)$, $(z) = (x^{2^n})$, we obtain a covariant of F in the narrow sense used in these lectures. In particular,

$$(29) \qquad K = \begin{vmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}, \qquad M = \begin{vmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ x_1^4 & x_2^4 & x_3^4 \end{vmatrix}$$

are formal covariants of F. While the discriminant Δ, given by (15), is a formal invariant, (16) is not. But

$$(30) \qquad A + \Delta + 1 \equiv \alpha \pmod{2},$$

$$(31) \qquad \alpha = \Sigma a_i b_i + \Sigma a_i^2 + a_1 a_2 + a_1 a_3 + a_2 a_3,$$

α being a formal invariant of F. By (23), the B's are contragredient to the x's and hence to the a's, so that

$$(32) \qquad \Delta_1 = \Sigma a_i B_i = \Sigma a_i b_i^2 + \Sigma a_i a_j^2 + a_1 a_2 a_3$$

is a formal invariant. For integral values of a_i, b_i,

$$(33) \qquad \Delta_1 \equiv \Delta \equiv \Sigma a_i(\beta_i + 1) \pmod{2}.$$

Any form with undetermined integral coefficients $c_1, c_2, \cdots$, taken modulo 2, has, by (21) of Lecture I, the invariant $(c_1 + 1)(c_2 + 1) \cdots$. Thus (16) is an invariant of (7) and hence of F. Likewise from (19) and F itself, we obtain the invariants

$$(34) \qquad J = \beta_1\beta_2\beta_3, \quad AJ = A\Pi(b_i + 1).$$

In (6) we made use geometrically of

$$(35) \qquad \lambda = u_1 x_1 + u_2 x_2 + u_3 x_3.$$

Now $F + t\lambda^2$ is congruent modulo 2 to the quadratic form derived from F by replacing each b_i by $b_i + tu_i^2$. Making this replacement in Δ, we see that the coefficient of t is congruent to κ^2, where

$$(36) \qquad \kappa = a_1 u_1 + a_2 u_2 + a_3 u_3$$

is therefore a formal invariant* of F and λ. Making the same

* Since (36) is a contravariant of F, $\Sigma a_i(\partial C/\partial x_i)$ is a covariant of F if C is. Taking Q_2, Q_1, L as C, we get K, M, Δ, respectively.

As to the classes, we saw in § 4 that, if F is not the square of a linear function (i. e., not reducible to x_1^2 or 0), it can be transformed into $x_1x_2 + b_1x_1^2 + b_2x_2^2 + \Delta x_3^2$ and hence into one of the first three classes of the table. By means of the relations

$$(44)\quad \Delta A \equiv 0,\quad \Delta J \equiv 0,\quad \Delta^2 \equiv \Delta,\quad A^2 \equiv A,\quad J^2 \equiv J \qquad (\text{mod } 2),$$

any polynomial in Δ, A, J equals a linear function of

$$(45)\qquad 1,\quad \Delta,\quad A,\quad J,\quad AJ.$$

These are linearly independent since there are five classes.

10. *Leader of a Covariant of F.*—Let S be the coefficient of x_3^ω in a covariant of order ω of F. Writing (14) in the form

$$(46)\quad F = f + lx_3 + b_3x_3^2,\quad f = b_1x_1^2 + a_3x_1x_2 + b_2x_2^2,\quad l = a_2x_1 + a_1x_2,$$

we see that the leader S is a function of b_3 and the invariants of the pair of forms f and l under the linear group on x_1, x_2.

In the modular covariants forming a fundamental system for f (§ 13 of Lecture III), we replace x_1 by a_1 and x_2 by a_2 and obtain a fundamental system of modular invariants of the pair f and l:

$$(47)\quad a_3,\quad \alpha_1\alpha_2,\quad q = b_1b_2 + (b_1 + b_2)\alpha_3,\quad j = (b_1 + a_3)a_1 + (b_2 + a_3)a_2,$$

where $\alpha_i = a_i + 1$. By means of the relations

$$(48)\qquad \alpha_1\alpha_2 j \equiv 0,\quad qj \equiv j + a_3 j \qquad (\text{mod } 2),$$

any polynomial in the four functions (47) can be reduced to a linear combination of

$$(49)\quad 1,\ a_3,\ q,\ a_3q,\ \alpha_1\alpha_2,\ \alpha_1\alpha_2a_3,\ \alpha_1\alpha_2q,\ \alpha_1\alpha_2a_3q,\ j,\ a_3j.$$

These form a complete set of linearly independent* invariants of f, l.

---

* Instead of verifying as usual that these 10 functions are linearly independent, we may deduce that result from the fact that there are 10 classes:

$$l = x_1,\quad f = a_3x_1x_2 + \alpha_3x_2^2 \quad \text{or} \quad qx_1^2 + a_3x_1x_2 + a_3x_2^2,$$

$$l \equiv 0,\quad f = x_1^2 + x_1x_2 + x_2^2,\quad x_1x_2,\quad x_1^2 \quad \text{or} \quad 0.$$

Since (47) characterize the classes, they form a fundamental system.

are contragredient with a_1, a_2, a_3 and hence with x_1, x_2, x_3, and therefore cogredient with u_1, u_2, u_3. Thus (39) yields the formal covariant

$$(41') \qquad W' = \Sigma(B_1B_2 + B_1^2 + B_2^2)\xi_3^2 + \Sigma B_1^2\xi_2\xi_3.$$

From this or (39′), we obtain the modular covariant

$$(41) \qquad W = \Sigma(\beta_1\beta_2 + 1)\xi_3^2 + \Sigma(\beta_1 + 1)\xi_2\xi_3.$$

In these notations (29) become

$$(42) \qquad K = \Sigma a_i\xi_i, \quad M = \Sigma a_1\xi_1(x_2^2 + x_2x_3 + x_3^2).$$

Finally, by (16) of Lecture III, we have the universal covariants

$$(43) \quad L_3 = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^4 & x_2^4 & x_3^4 \end{vmatrix}, \qquad \begin{aligned} Q_1 &= \Sigma x_1^4x_2^2 + \Sigma x_1^4x_2x_3 + x_1^2x_2^2x_3^2, \\ Q_2 &= \Sigma x_1^4 + \Sigma x_1^2x_2^2 + x_1x_2x_3\Sigma x_1. \end{aligned}$$

The covariant line $L \equiv 0$ of a non-degenerate conic $F \equiv 0$ is determined by the three (collinear) diagonal points of the complete quadrangle having as its vertices the apex (a) and the three intersections of $F \equiv 0$ with its covariant cubic curve $K \equiv 0$.

FUNDAMENTAL SYSTEM OF COVARIANTS OF THE TERNARY FORM F, §§ 9–32

9. *Invariants of F.*—A fundamental system of invariants of F is given by Δ, A, J. It suffices to prove that they completely characterize the classes of forms F under the group of all ternary linear transformations with integral coefficients modulo 2. This is evident from the following table

Class	Δ	A	J	L
$x_1x_2 + x_3^2$	1	0	0	$x_1 + x_2 + x_3$
$x_1x_2 + x_1^2 + x_2^2$	0	0	1	0
x_1x_2	0	0	0	$x_1 + x_2$
x_1^2	0	1	0	x_1
0	0	1	1	0

where $R \equiv b_3 + a_2$ is the increment to b_1 in (50). Set

(56) $\sigma = pa_3b_1 + ra_3 + sb_1 + t$ ($p, \cdots, t$ independent of a_3, b_1).

Let the substitution (50) replace σ by σ'. Then

(57) $$\sigma' - \sigma = pRa_3 + pa_1b_1 + pa_1R + ra_1 + sR.$$

This is zero for every a_3, b_1 if and only if

(58) $$pR \equiv 0, \quad pa_1 \equiv 0, \quad ra_1 \equiv sR \pmod 2.$$

For $p = m_2 + \cdots$, $pa_1 \equiv 0$ gives $m_3 \equiv 0$, $m_4 \equiv m_2$. Then $pR \equiv 0$ gives $m_2 \equiv 0$, $mb_3 \equiv 0$, whence $m \equiv 0$. Thus $\sigma = m_1a_3$, so that $m_1 \equiv 0$. Hence *the leader of a covariant of F has the form*

(59) $$I + b_3I_1 + c\alpha_1\alpha_2 + d\alpha_1\alpha_2b_3,$$

where I and I_1 are invariants, c and d are constants.

Covariants Whose Leaders are Not Zero, §§ 11–19

11. Consider a covariant of odd order ω:

(60) $$C = Sx_3^\omega + S_1x_3^{\omega-1}x_1 + S_2x_3^{\omega-2}x_1^2 + \cdots.$$

If S_1' is derived from S_1 by the substitution (50), then, by (51),

(61) $$S_1' \equiv S_1 + \omega S \equiv S_1 + S \pmod 2.$$

Give S_1 the notation (56). Then S is given by (57) and has no term with the factor a_3b_1. Now a_3b_1 enters no term of (59) except J and AJ of I and* b_3J of b_3I_1, and in these is multiplied by

(62) $$b_3\alpha_1 + \alpha_1\alpha_2, \quad \alpha_1\alpha_2(b_2+1)(b_3+1), \quad b_3\alpha_1a_2,$$

respectively. Since the latter are linearly independent, neither J nor AJ occurs in the I, I_1 of the leader (59). Also, A and $\alpha_1\alpha_2$ occur only in the combinations $A + 1$, $\alpha_1\alpha_2 + 1$, since (57) has no constant term. The coefficients of x_3^ω in L^ω, AL^ω, $(A + \Delta)L^\omega$ are respectively

(63) $$b_3 + \alpha_1\alpha_2 + 1, \quad Ab_3, \quad \Delta + \Delta b_3 + b_3\alpha_1\alpha_2,$$

* AJ is not retained in I_1, since $b_3AJ \equiv 0$, AJ being (34).

Hence S is a linear combination of the functions (49) and their products by b_3. Moreover, S must remain unaltered modulo 2 when a_3 and b_1 are replaced by

$$(50)\qquad a_3' \equiv a_3 + a_1, \quad b_1' \equiv b_1 + b_3 + a_2,$$

which are the only altered coefficients of the form obtained from F by the transformation

$$(51)\qquad x_1 \equiv x_1', \quad x_2 \equiv x_2', \quad x_3 \equiv x_3' + x_1' \qquad (\text{mod } 2).$$

Both requirements are evidently met by the functions

$$(52)\qquad 1, \quad \alpha_1\alpha_2, \quad b_3, \quad b_3\alpha_1\alpha_2$$

and any invariant of F. We find that

$$A = \alpha_1\alpha_2(a_3 + 1), \quad \Delta = \alpha_1\alpha_2a_3 + j + a_3b_3 + a_3,$$

$$(53)\qquad J = \alpha_1\alpha_2(a_3 + 1)(b_3 + 1) + b_3j + a_3b_3j + b_3q + \alpha_1\alpha_2q,$$

$$AJ = \alpha_1\alpha_2(a_3 + 1)(b_3 + 1)(q + 1).$$

From these and their products by b_3, we see that

$$(54)\qquad AJ, \quad b_3J, \quad J, \quad b_3\Delta, \quad b_3A, \quad \Delta, \quad A$$

contain the respective terms

$$b_3\alpha_1\alpha_2a_3q, \quad b_3\alpha_1\alpha_2q, \quad \alpha_1\alpha_2q, \quad b_3j, \quad b_3\alpha_1\alpha_2a_3, \quad j, \quad \alpha_1\alpha_2a_3,$$

while no one involves an earlier one of these terms. Hence any linear combination of the functions (49) and their products by b_3 is a linear combination of the functions (52), (54) and

$$(55)\qquad a_3, \quad b_3a_3, \quad q, \quad b_3q, \quad a_3q, \quad b_3a_3q, \quad a_3j, \quad b_3a_3j, \quad \alpha_1\alpha_2a_3q.$$

A linear combination of the latter is of the form

$$\sigma = m_1a_3 + m_2q + m_3a_3q + m_4a_3j + m\alpha_1\alpha_2a_3q,$$

where $m_1, \cdots, m_4$ are linear functions of b_3, while m is a constant. The coefficient of a_3b_1 is seen to be

$$p = m_2 + m_3b_2 + m_4a_1 + mb_2\alpha_1(R + b_3 + 1),$$

combination of the covariants (65), while C' is a covariant whose leader is an invariant. For $\omega = 2$,

$$C' = Sx_3^2 + S_1x_3x_1 + Sx_1^2 + x_2\phi.$$

This is transformed by (51) into a function having S_1 as the coefficient of $x_1'^2$. Since S is an invariant, $S_1 = S$. Thus every coefficient of C' equals S. Then (51) transforms C' into a function in which the coefficient of $x_1'x_2'$ is zero, so that $S = 0$. Hence *every quadratic covariant is a linear function of*

(66) $$F, \quad AF, \quad \Delta F, \quad JF, \quad L^2, \quad \Delta L^2.$$

14. There remains the more difficult case of covariants (60) of order $\omega = 4n + 2 > 2$. If S_i' is the function obtained from S_i by the substitution (50), then

(67) $$S_1' = S_1, \quad S_2' = S + S_1 + S_2.$$

Now S_1 is unaltered also by the substitutions (22) and

(68) $$a_3' \equiv a_3 + a_2, \quad b_2' \equiv b_2 + b_3 + a_1 \qquad (\text{mod } 2),$$

induced on the coefficients of F by the transformations (8) and

(69) $$x_1 = x_1', \quad x_2 = x_2', \quad x_3 = x_3' + x_2'.$$

15. *A fundamental system of invariants of F, under the group Γ generated by the transformations* (8), (51) *and* (69), *is given by* $A, \Delta, J, a_2, b_3, a_1\alpha_2$ *and*

(70) $$\beta = b_1(b_3 + \alpha_2).$$

It suffices to prove that these seven functions, which are evidently invariant under Γ, completely characterize the classes of forms F under Γ. There are six cases.

(i) $b_3 \equiv a_2 \equiv 1$. Replacing x_1 by $x_1 + a_1x_2$ and x_3 by $x_3 + a_3x_2$, we get

$$F = \beta x_1^2 + \Delta x_2^2 + x_3^2 + x_1x_3.$$

(ii) $b_3 \equiv 1,\ a_2 \equiv 0,\ a_1\alpha_2 \equiv 1$. Replacing x_3 by $x_3 + a_3x_1$, we get

$$F = \Delta x_1^2 + b_2x_2^2 + x_3^2 + x_2x_3.$$

where L is the linear covariant (19). After subtracting from C a linear combination of these three covariants, we may set

$$S = m_1(A + 1) + m_2\Delta + m_3 b_3 + m b_3 \alpha_1 \alpha_2.$$

Since $\beta_3 b_3 \alpha_1 \alpha_2 \equiv 0$, $\Delta J \equiv 0$, the leader of the covariant JC is

$$JS = m_1 AJ + m_1 J + m_3 b_3 J.$$

Hence $m_1 \equiv m_3 \equiv 0$. The coefficient of a_3 in S is now $m_2(a_1 a_2 + b_3)$ and must vanish for $b_3 \equiv a_2$ since it is of the form pR by (57). Hence $m_2 \equiv 0$. Thus $S = m b_3 \alpha_1 \alpha_2$. For $\omega > 1$, $mFL^{\omega-2}$ has this same leader. For $\omega = 1$,

$$C = m(b_3 \alpha_1 \alpha_2 x_3 + b_1 \alpha_2 \alpha_3 x_1 + b_2 \alpha_1 \alpha_3 x_2),$$

which satisfies (61) only when $m = 0$. Hence *every linear covariant is a linear function of* L, AL, ΔL; *every covariant of odd order* $\omega > 1$ *differs from a linear combination of* L^ω, AL^ω, ΔL^ω, $FL^{\omega-2}$ *by a covariant whose leader is zero.*

12. In the covariants of order $4n$

(64) $$IQ_2^n, \quad IF^{2n}, \quad L^{4n}, \quad F^{2n-1}L^2 \quad (I \text{ an invariant}),$$

the coefficients of x_3^{4n} are respectively

$$I, \quad b_3 I, \quad b_3 + \alpha_1\alpha_2 + 1, \quad b_3\alpha_1\alpha_2.$$

Linear combinations of these give every leader (59). Hence *every covariant of order* $4n$ *differs from a linear combination of the covariants* (64) *by a covariant whose leader is zero.*

13. In the covariants of order $\omega = 4n + 2$

(65) $$IQ_2^nF, \quad Q_2^nL^2, \quad \Delta Q_2^nL^2 \quad (I \text{ an invariant}),$$

the coefficients of x_3^ω are respectively

$$b_3 I, \quad b_3 + \alpha_1\alpha_2 + 1, \quad \Delta + b_3(\Delta + \alpha_1\alpha_2 a_3).$$

The sum of the third function and $b_3(A + \Delta)$ is $\Delta + b_3\alpha_1\alpha_2$. Hence any covariant C is of the form $P + C'$, where P is a linear

$$(71)\quad A\beta=b_1(b_3+1)A,\quad b_3\Delta=b_1b_3a_1+\cdots,\quad a_2b_3\Delta=b_1b_3a_1a_2+\cdots,$$

$$J=b_1b_2b_3+\cdots,\quad a_2J=b_1b_2b_3a_2+\cdots,$$

$$b_3J=b_1b_2b_3(a_1a_2+a_1+a_2)+\cdots,\quad AJ=b_1b_2b_3A+\cdots.$$

These are linearly independent since the first eight do not involve b_1, while all the terms with the factor b_1 in the next seven are given explicitly, likewise all with the factor $b_1b_2b_3$ in the last four. Hence *the* 19 *functions* (71) *form a complete set of linearly independent invariants of F under the group* Γ.

17. Hence, in § 14, S_1 is a linear combination of the functions (71). By (67_2), $S+S_1$ is of the form (57) if S_2 be denoted by (56). Now a_3b_1 occurs in J, AJ, b_3J, a_2J, $A\beta$, but in no further function (71). In the first three, a_3b_1 is multiplied by the linearly independent functions (62), respectively; in the last two by $b_3\alpha_1a_2$ and $\alpha_1\alpha_2(b_3+1)$, whose sum is congruent to the first function (62). Hence the part of $S+S_1$ involving $J, \cdots, A\beta$ is a linear combination of

$$(72)\qquad (b_3+a_2)J=b_1b_2b_3a_1\alpha_2+b_2b_3a_1\alpha_2\alpha_3,$$

$$(73)\qquad J+b_3J+A\beta=(b_3+1)(b_1b_2\alpha_1\alpha_2+b_2A+A).$$

But b_1 occurs in just six of the functions (71) other than the five just considered. Thus the factor pa_1 of b_1 in (57) is a linear combination of the coefficients of b_1 in (72), (73), β, $a_2\beta$, Δ, $a_2\Delta$, $b_3\Delta$, $a_2b_3\Delta$. Now a_1 is a factor of the coefficients of b_1 in all except the second, third and fourth, while in these the coefficients are

$$(b_3+1)b_2\alpha_1\alpha_2,\quad b_3+a_2+1,\quad a_2b_3$$

and are linearly independent. Hence (73), β, $a_2\beta$ do not occur in $S+S_1$. By (57), the latter has no constant term and hence involves 1, A only in the combination $A+1$. This cannot occur since the total coefficient of a_3 must be of the form pR and hence vanish for $b_3\equiv a_2$. At the same time we see that the sum of the constant multipliers of Δ, $a_2\Delta$, $b_3\Delta$, $a_2b_3\Delta$ is zero modulo 2. Hence $S+S_1$ is a linear combination of the functions

If $\Delta \equiv 0$, then $b_2 \equiv J$. If $\Delta \equiv 1$, we replace x_1 by $x_1 + b_2x_2$ and get

$$x_1^2 + x_3^2 + x_2x_3.$$

(iii) $b_3 \equiv 1$, $a_2 \equiv a_1\alpha_2 \equiv 0$. Replacing x_3 by $x_3 + b_1x_1 + b_2x_2$, we get

$$x_3^2 + \Delta x_1x_2.$$

(iv) $b_3 \equiv 0$, $a_2 \equiv 1$. After replacing x_3 by $x_3 + a_3x_2$, we obtain a form with also $a_3 \equiv 0$. Taking this as F, and replacing x_1 by $x_1 + a_1x_2$, we get

$$b_1x_1^2 + \Delta x_2^2 + x_1x_3.$$

Replacing x_3 by $x_3 + b_1x_1$, we get $\Delta x_2^2 + x_1x_3$.

(v) $b_3 \equiv a_2 \equiv 0$, $a_1\alpha_2 \equiv 1$. Replacing x_3 by $x_3 + a_3x_1 + b_2x_2$, we get

$$\beta x_1^2 + x_2x_3.$$

(vi) $b_3 \equiv a_2 \equiv a_1\alpha_2 \equiv 0$. Then F is the binary form f in (46). The effective part of Γ is now the subgroup Γ_1 generated by (8). Now

$$\beta \equiv b_1, \quad A + 1 \equiv a_3, \quad J \equiv B + (b_1 + 1)\alpha_3, \quad B = b_2(b_1 + \alpha_3).$$

These seminvariants b_1, a_3, B of f completely characterize the classes of forms f under Γ_1. For, if $a_3 \equiv b_1$,

$$f = b_1x_1^2 + Bx_2^2 + b_1x_1x_2;$$

while if $a_3 \equiv b_1 + 1$, we replace x_1 by $x_1 + b_2x_2$ and get

$$b_1x_1^2 + (b_1 + 1)x_1x_2.$$

16. The number of classes of forms F in the respective cases (i)–(vi) is 4, 3, 2, 2, 6. Hence there are exactly 19 linearly independent invariants of F under the group Γ. As these we may take

$$1, \quad a_2, \quad a_1\alpha_2, \quad A, \quad b_3, \quad b_3a_2, \quad b_3a_1\alpha_2, \quad b_3A,$$

$$\Delta = b_1a_1 + \cdots, \quad a_2\Delta = b_1a_1a_2 + \cdots,$$

$$(71) \qquad \beta = b_1(b_3 + \alpha_2), \quad a_2\beta = b_1b_3a_2,$$

Hence $C - Q_2^{n-1}E$ has the leader zero. *Any covariant of order* $\omega = 4n + 2 > 2$ *differs from a linear combination of the covariants* (65) *and* $Q_2^{n-1}E$ *by a covariant whose leader is zero.*

19. *Regular and Irregular Covariants; Rank.*—A covariant shall be called regular or irregular according as it has not or has the factor L_3, given by (43). The quotient of an irregular covariant by L_3 is a covariant. Hence the determination of all irregular covariants reduces to that of the regular covariants. If a covariant has a linear factor it has as a factor each of the seven ternary linear functions incongruent modulo 2, whose product is L_3. Hence a regular covariant has a non-vanishing component involving only x_1, x_3. In a regular covariant C without terms x_i^ω (i. e., with leader zero), this component has the factors x_1, x_3 and (by the covariant property) also $x_1 + x_3$. The product of these three linear factors was denoted by ξ_2 in (40). Let ξ_2^m be the highest power of ξ_2 which is a factor of the component and let n be the degree of the quotient in the x's. Then C may be given the notation

$$(76) \qquad R_{m,\,n} = \sum_{i=1}^{3} f_i \xi_i^m + x_1 x_2 x_3 \phi,$$

where, if $n = 0$, f_2 is a function of the a's and b's not identically zero, while, if $n > 0$, f_2 is a function also of x_1, x_3 in which the coefficients of x_1^n and x_3^n are not zero; f_1 is a function of x_2, x_3; f_3 of x_1, x_2.

The regular covariant (76) shall be said to be of *rank m*. In an irregular covariant the component free of x_2 is zero and hence is divisible by an arbitrary power of ξ_2; it is proper and convenient to say that an irregular covariant is of infinite rank.

Any covariant of rank zero differs from one of rank greater than zero by a polynomial in the known covariants

$$(77) \qquad A, \quad \Delta, \quad J, \quad F, \quad L, \quad Q_2.$$

This is a consequence of the theorems in §§ 11–18, where the polynomial is given explicitly. Any product, of order ω in the

a_2, b_3, b_3a_2, $a_1\alpha_2$, and the last six in (74) below. Like (57), this combination must vanish for $a_1 \equiv 0$, $b_3 \equiv a_2$. Since all but the first three of the ten functions then vanish, the sum of the multipliers of these three must be zero modulo 2. Hence $S + S_1$ is a linear combination of

$$(74)\qquad \begin{array}{l} b_3 + a_2, \quad a_2(b_3 + 1), \quad a_1\alpha_2, \quad b_3a_1\alpha_2, \quad b_3A, \\ \Delta\alpha_2, \quad \Delta(b_3 + 1), \quad \Delta(a_2b_3 + 1), \quad (b_3 + a_2)J. \end{array}$$

18. Without altering the invariant S, we may simplify S_1 by subtracting from C constant multiples of $L^{4n-1}K$ and its product by Δ, where K is given by (29), and hence delete $a_2(b_3 + 1)$ and $\Delta(a_2b_3 + 1)$ from the terms (74) of S_1. Then

$$S_1 = S + m\Delta\alpha_2 + m_1\Delta(b_3 + 1) + m_2(b_3 + a_2)J$$
$$+ m_3(b_3 + a_2) + m_4a_1\alpha_2 + m_5b_3a_1\alpha_2 + m_6b_3A.$$

The coefficient T of $x_3^{\omega-1}x_2$ in C is obtained from S_1 by applying the substitution $(a_1a_2)(b_1b_2)$ induced by (x_1x_2). In view of the transformation (8), we see that $T' = T + S_1$, where T' is derived from T by (22). Hence

$$S = (m + m_1)\Delta + m_1b_3\Delta + m_2b_3J$$
$$+ (m_4 + m_5b_3)(a_1a_2 + a_1 + a_2) + m_3b_3 + m_6b_3A.$$

Let Σ be the sum of the second member and the function obtained from it by the substitution $(a_2a_3)(b_2b_3)$. Thus $\Sigma \equiv 0$. Taking $b_3 \equiv b_2$, we get $m_4 \equiv m_5 \equiv 0$. Then

$$\Sigma = (b_2 + b_3)I, \quad I = m_1\Delta + m_2J + m_3 + m_6A.$$

Applying to Σ the substitution (68), we get $(b_2 + a_1)I = 0$. Applying $(a_1a_3)(b_1b_3)$ to the latter, we get $(b_2 + a_3)I = 0$. Adding, we get $(a_1 + a_3)I = 0$. Applying (50), we see that $a_3I = 0$. Then each $a_iI = 0$, so that $I = gA$, where g is a constant. By $\Sigma = 0$, $g = 0$. Thus m_1, m_2, m_3, m_6 are zero. Hence $S = m\Delta$, $S_1 = m\Delta a_2$. But

$$(75)\qquad E = F(L^4 + \Delta F^2) + (\Delta + A)L^6 = \Delta x_3^6 + \cdots.$$

since a term previously replaced is not introduced later. Thus f_3 is a linear combination of these seven functions, a_3, q, a_3q, and

$$\alpha_1\alpha_2, \quad \alpha_1\alpha_2 q, \quad b_3, \quad b_3a_3, \quad b_3q, \quad b_3\alpha_1\alpha_2, \quad b_3\alpha_1\alpha_2 q, \quad b_3 j, \quad b_3a_3 j.$$

Give to any linear function $m_1\alpha_1\alpha_2 + \cdots$ of these the notation

$$\sigma = \alpha a_1 b_3 + \beta a_1 + \gamma b_3 + \delta.$$

Call e the increment $b_1 + a_2$ to b_3 in (80) and employ e to eliminate b_1. Then σ is unaltered by (80) if and only if

$$\alpha e \equiv 0, \quad \alpha a_3 \equiv 0, \quad \beta a_3 \equiv \gamma e \qquad (\text{mod } 2).$$

Since b_3 does not occur in q or j, nor a_1 in q, we have

$$\alpha = m_6\alpha_2 + m_7\alpha_2 q + m_8(e + a_2 + a_3) + m_9 a_3(e + a_2 + a_3).$$

Thus $\alpha e \equiv 0$ gives $m_6 \equiv m_7 \equiv 0$, $m_8 \equiv m_9$. Then $\alpha a_3 \equiv 0$ gives $m_9 \equiv 0$. Now

$$\beta = m_1\alpha_2 + m_2\alpha_2 q, \quad \gamma = m_3 + m_4 a_3 + m_5 q,$$

and $\beta a_3 \equiv \gamma e$ readily gives $\sigma \equiv 0$. *Any function of b_3 and the invariants* (49) *of f and l, which is unaltered by* (80), *is a linear combination of the ternary invariants* (45) *and a_3, q, a_3q, $a_3\Delta$, a_3J, qA.*

21. For $n = 0$ and m even, there exists a covariant (76) in which f_3 is any function specified in the preceding theorem. For, if I is any ternary invariant, $IQ_1^{m/2}$ has $f_3 = I$. By (42) and (41), K^m and $W^{m/2}$ are of the form (76) with $f_3 = a_3$ and $\beta_1\beta_2 + 1$, respectively; they may be multiplied by any invariant. By (19) and (47), we have

$$(81) \quad \beta_1\beta_2 + 1 = q + \alpha_3\Delta + A + 1, \quad a_3 q = \alpha_3\Delta + q\Delta + a_3 J.$$

Hence we obtain q, then qA, $q\Delta$, and therefore a_3q. *Any covariant with $n = 0$, m even, differs by an irregular covariant from a linear function of*

$$IQ_1^{m/2}, I_1K^m, I_2W^{m/2} \quad (I = 1, \Delta, A, J, AJ;\ I_1 = 1, \Delta, J;\ I_2 = 1, \Delta, A).$$

x's, of powers of the covariants (77) can be reduced by means of the syzygies

$$JL = 0, \quad AL^2 = AF, \quad (\Delta + A + J + 1)(FL + K) = 0,$$

$$(78) \quad AK = 0, \quad FL^2 + (A + \Delta)L^4 + \Delta F^2 + \Delta Q_2 = LK,$$

$$F^3 + Q_2 F = L^3 K + (\Delta + J)K^2 + (\Delta + 1)LG + (A + 1)Q_1,$$

to a sum of covariants of order ω given in §§ 11–18 and a linear function, with covariant coefficients, of K, Q_1 and

$$\begin{aligned} G = Q_2 L + L^5 = \Sigma\xi_2[\beta_3(\beta_1 + 1)x_3^2 + (\beta_1\beta_3 + 1)x_3 x_1 \\ (79) \qquad + \beta_1(\beta_3 + 1)x_1^2] + x_1x_2x_3[(\beta_1 + \beta_2 + \beta_3 + 1) \\ \times (x_1x_2 + x_1x_3 + x_2x_3) + \Sigma(\beta_i + 1)x_i^2]. \end{aligned}$$

Here G and K, given by (42), are of rank 1, while $Q_1 = \xi_2^2 + x_2(\ )$ is of rank 2. As this theorem is not presupposed in what follows, its proof is omitted. However, it led naturally to the important relations (75) and (79) and showed that no new combinations of the covariants (77) of rank zero yield covariants of rank > 0, a fact used as a guide in the investigation of the latter covariants.

Regular Covariants R_{m0}, §§ 20–22

20. A separate treatment is necessary for covariants (76) with $n = 0$. Then each f_i is a function of the coefficients a_j, b_j. Since the factor ξ_3^m of the part $f_3\xi_3^m$ of R_{m0} free of x_3 is unaltered by every linear transformation on x_1 and x_2, f_3 is a linear combination of the functions (49) and their products by b_3. Also, f_3 must be unaltered by

$$(80) \quad x_1 = x_1' + x_3': \quad a_1' = a_1 + a_3, \quad b_3' = b_3 + b_1 + a_2.$$

Both conditions are evidently satisfied by the ternary invariants and by a_3 and q, in (47). In view of (53), we may employ

$$AJ, \quad J, \quad a_3\Delta, \quad \Delta, \quad a_3J, \quad qA, \quad A$$

to replace in turn

$$b_3\alpha_1\alpha_2a_3q, \quad b_3\alpha_1\alpha_2a_3, \quad a_3j, \quad j, \quad a_3b_3q, \quad \alpha_1\alpha_2a_3q, \quad \alpha_1\alpha_2a_3,$$

Taking $b_3 \equiv a_2$, we see that $n_1 \equiv n_2 \equiv d \equiv 0$. Thus $f_1 = cq'$. By (57) for $a_1 \equiv 0$, $b_3 \equiv a_2$, we get $c \equiv 0$. *Any covariant with* $n = 0$ *and* m *odd differs by an irregular covariant from a linear function of* K^m, ΔK^m, JK^m *and, if* $m > 1$, $KW^{(m-1)/2}$.

Covariants of Rank Unity, §§ 23–26

23. Henceforth let $m > 0$, $n > 0$ in (76) and set

$$f_2 = Sx_3^n + S_1x_3^{n-1}x_1 + S_2x_3^{n-2}x_1^2 + \cdots \quad (S \neq 0). \tag{84}$$

Since S is unaltered by the group Γ of § 15, it is a linear combination of the functions (71). We may omit the functions $a_2(b_3 + 1)$ and $\Delta a_2(b_3 + 1)$, since K^mL^n is of the form (76) with $S = a_2(b_3 + 1)$. Thus

$$S = I + a_2I_1 + b_3I_2 + k_1a_1\alpha_2 + k_2b_3a_1\alpha_2 + k_3\beta + k_4a_2\beta + k_5A\beta, \tag{85}$$

where I is any invariant, I_1 a linear function of $1, \Delta, J$; I_2 one of $1, A, \Delta, J$; while $\beta = b_1(b_3 + \alpha_2)$.

First, let $m = 1$. If T and B are the coefficients of x_2^n in f_3 and f_1, transformation (51) replaces the covariant (76) by a function in which, by (82), the coefficient of $x_1'x_2'^{n+2}$ is

$$T + B = T', \tag{86}$$

where T' is derived from T by the induced substitution (50). But T is obtained from S by the interchange [23] of subscripts, and B from T by [13]. We thus find by (86) that

$$\begin{aligned} I = b_2I_2 + (k_1 + k_2b_2)(a_1 + a_3\alpha_1) \\ + k_3(a_1b_1 + a_2b_2 + a_3b_3 + a_1a_2 + a_2\alpha_3) \\ + k_4b_2(a_1b_1 + a_3b_3 + a_1a_2 + a_2a_3). \end{aligned}$$

Let Σ be the sum of the second member and the function obtained by applying $(a_2a_3)(b_2b_3)$ to it. In $\Sigma = 0$, set $b_2 = b_3$; we get

$$\{k_1 + k_3 + b_3(k_2 + k_4)\}(a_2 + a_3)\alpha_1 = 0, \quad k_3 \equiv k_1, \quad k_4 \equiv k_2.$$

Then $\Sigma = 0$ may be written in the form

$$(b_2 + b_3)\lambda = 0, \quad \lambda = I_2 + k_2(\Delta + A + 1).$$

22. For $n = 0$ and m odd, we may delete the terms $a_3 I_1$ from f_3 by use of $I_1 K^m$. First, let $m = 1$ and apply transformation (51); we get

$$\xi_1 = \xi_1' + \xi_3', \quad \xi_2 = \xi_2', \quad \xi_3 = \xi_3',$$
$$(82) \qquad R' = f_1\xi_1' + f_2\xi_2' + (f_1 + f_3)\xi_3' + (x_1'x_2'x_3' + x_1'^2 x_2')\phi.$$

Thus $\phi = 0$. Since $f_3 = I + I_2 q$, condition $f_1 + f_3 = f_3'$ gives

$$I = I_2(a_1 b_1 + \alpha_2 b_2 + a_3 b_3 + a_2\alpha_3 + a_1 a_2).$$

Add to this the relation obtained by permuting the subscripts 1, 2. Thus

$$0 = I_2(b_1 + b_2 + a_2\alpha_3 + a_1\alpha_3).$$

The increment under (22) is $I_2(b_1 + a_3 + a_2\alpha_3) = 0$. Now I_2 is of the form $x + y\Delta + zA$, where x, y, z are constants. From the terms in $b_1 b_2$, we get $y = 0$. Then $x = z = 0$. *The only covariants are therefore* $I_1 K$.

Second, let $m > 1$. Then $KW^{(m-1)/2}$ is of the form (76) with $f_3 = a_3 q + a_3$, by (81_1). Hence we may set

$$f_3 = I + cq + dqA \qquad (c, d \text{ constants}).$$

In R given by (76), let g denote the coefficient of

$$(83) \qquad x_1 x_2 x_3 \cdot x_2^m x_3^{2m-3}.$$

In the function derived from R by the transformation (51), the term corresponding to (83) has the coefficient $g + f_1$, since by (82) the ξ_i parts contribute only one such term, that from $f_1 \xi_1'^{m-1} \xi_3'$. Now

$$f_1 = I + cq' + dq'A, \quad q' = b_2 b_3 + (b_2 + b_3)\alpha_1.$$

When g is given the notation (56), $g' - g = f_1$ is the function (57). But $a_3 b_1$ occurs in f_1 only in J and AJ and in them with the linearly independent multipliers (62). Hence

$$I = n_1(A + 1) + n_2\Delta.$$

The coefficient of a_3 in f_1 is now

$$n_1\alpha_1\alpha_2 + n_2(a_1 a_2 + b_3) + dq'\alpha_1\alpha_2 = p(b_3 + a_2).$$

$x_1x_2x_3 \cdot x_2x_3^{n-1}$ in R', we have

$$B_n + B_{n-1} + l = l'.$$

For l given by (56), $B_n + B_{n-1}$ is given by (57). By the coefficient of a_3b_1, we get $t_4 \equiv 0$. The coefficient of a_3 must vanish for $b_3 = a_2$. Hence

$$k_1\alpha_1 + (k_2 + t_3)\alpha_1a_2 + k_5\alpha_1\alpha_2b_2 = 0, \quad k_1 \equiv k_5 \equiv 0, \quad t_3 \equiv k_2,$$

$$S = k_2b_3(\Delta + A + 1 + a_1\alpha_2 + a_2b_1).$$

The coefficient of k_2 equals that of $\xi_2x_3^n$ in

$$GFQ_2^{\nu-1} + \Delta KL^n + \Delta KQ_2^\nu.$$

Any covariant with $m = 1$, $n = 4\nu$, *differs from one with* $m \geqq 2$ *by a linear function of* KL^n, ΔKL^n, IKQ_2^ν, $GFQ_2^{\nu-1}$ $(I = 1, \Delta, J)$.

25. For $m = 1$, $n = 4\nu + 2$, we may delete a_2I_1 from S, given by (87), by use of $I_1Q_2^\nu M$. The coefficient of $\xi_2x_3^n$ in $Q_2^\nu G$ is

$$d = \beta_3(\beta_1 + 1) = A + (b_1 + 1)(\alpha_1\alpha_2 + b_3) + b_3\alpha_2\alpha_3.$$

The coefficient of k_1 in S equals $d + a_2\Delta + a_2(b_3 + 1)$, the final term of which was reached in § 23, and $a_2\Delta$ above. The coefficients of k_5 and k_2 in S equal Ad and

$$b_1b_3(a_1+a_2)+b_3(a_2b_2+a_1a_3+a_2a_3+a_2)=\Delta d+a_2(J+1)+a_2(b_3+1),$$

respectively. *Any covariant with* $m = 1$, $n = 4\nu + 2$, *differs from one with* $m \geqq 2$ *by a linear function of* KL^n, ΔKL^n, $IQ_2^\nu G$, $I_1Q_2^\nu M$ $(I = 1, A, \Delta;\ I_1 = 1, \Delta, J)$.

For use in § 26, we replace $Q_2^\nu M$ by $Q_2^\nu FK$, noting that

$$M = (F + L^2)K \tag{88}$$

and that $Q^\nu L^2K$ differs from KL^n by a covariant of rank 2.

26. By the last four theorems, any covariant of rank 1 differs from one of rank $\geqq 2$ by $CK + DG$, where C and D are known covariants of rank zero. Taking as C_1 and D_1 arbitrary func-

As in § 18, $\lambda = 0$. Thus I_2 and I are the products of $\Delta + A + 1$ by k_2, k_1, so that

$$S = (k_1 + k_2 b_3)(\Delta + A + 1) + a_2 I_1 + k_1(b_1 b_3 + b_1\alpha_2 + a_1\alpha_2) \tag{87}$$
$$+ k_2 b_3(a_2 b_1 + a_1\alpha_2) + k_5 A(b_1 b_3 + b_1).$$

For n odd, S is the increment to S_1 under (50) and hence has no term containing $a_3 b_1$. If t is the coefficient of J in I_1, $a_3 b_1$ occurs in (87) only in $ta_2 J$ and in the final part, being multiplied by $ta_2\alpha_1 b_3$ and $k_5\alpha_1\alpha_2(b_3 + 1)$, respectively. Hence $t \equiv k_5 \equiv 0$. Since S is of the form (57), the coefficient of b_1 must vanish if $a_1 \equiv 0$. Thus

$$k_1(b_3 + \alpha_2) + k_2 b_3 a_2 = 0, \quad k_1 \equiv k_2 \equiv 0.$$

Now $S = a_2 I_1 = a_2(u + v\Delta)$ must vanish for $a_1 \equiv 0$, $b_3 \equiv a_2$ by (57); then $\Delta = a_2(b_2 + a_3)$, so that $u = v = 0$, $S = 0$. *Any covariant with $m = 1$ and n odd differs from one of rank > 1 by a linear function of KL^n, ΔKL^n.*

24. For $m = 1$, $n = 4\nu$, we may delete $a_2 I_1$ from (85) by use of $I_1 KQ_2^{\nu}$. Set $f_1 = Bx_2^n + \cdots + B_n x_3^n$. Then (51) replaces (76) by

$$R' = \xi_2[Sx_3^n + S_1 x_3^{n-1}x_1 + (S_1 + S_2)x_3^{n-2}x_1^2 + \cdots] + \xi_3 f_3$$
$$+ (\xi_1 + \xi_3)[B_n(x_3^n + x_3^{n-4}x_1^4 + \cdots)$$
$$+ B_{n-1}x_2(x_3^{n-1} + x_3^{n-2}x_1 + \cdots)] + (x_1x_2x_3 + x_1^2x_2)\phi'.$$

Since S_1 is the increment of S_2, it is a linear combination of the functions (74). By use of $L^{n-3}Q_1$, $L^{n-3}K^2$ and their products by A and Δ, we may, without disturbing S, delete from S_1

$$b_3 + \alpha_1\alpha_2 + 1, \quad Ab_3, \quad \Delta + b_3\Delta + b_3\alpha_1\alpha_2, \quad a_2(b_3 + 1), \quad a_2\Delta(b_3 + 1).$$

Hence we may set

$$S_1 = t_1(b_3 + a_2) + t_2 b_3 a_1\alpha_2 + t_3\Delta\alpha_2 + t_4(b_3 + a_2)J.$$

Applying $(a_1a_2)(b_1b_2)$ to S and S_1, we obtain B_n and B_{n-1}. Let l be the coefficient of $x_2x_3^{n-1}$ in ϕ. By the coefficient of

Such a term occurs in neither of the first two parts of R', since they are functions of only two variables. To obtain such a term from the third part of R', we must omit terms with the factor ξ_3^2 (and hence x_1^2) and take $(x_2x_3^2)^{2\mu}$ in $\xi_1^{2\mu}$, so as not to make the degree in x_2 too high. Hence if T be the coefficient of x_3^n in f_1, $g' \equiv g + T$. Now $(a_1a_2)(b_1b_2)$ replaces S by T. The resulting T must be of the form (57). By the coefficient of a_3b_1, $k_4 \equiv 0$; cf. (72). By the coefficient $k_3\alpha_1b_3$ of a_3, $k_3 \equiv 0$. Since $T \equiv 0$ for $a_1 \equiv 0$, $b_3 \equiv a_2$, we get $k_1 \equiv k_2$. Hence $S = k_1v$, where v is given by (90).

For $n = 1$, $f_2 = Sx_3 + S_1x_1$. Thus $S_1 = k_1v'$, where v' is derived from v by interchanging the subscripts 1 and 3. Then $S_1' \equiv S_1 + S$ gives $k_1 \equiv 0$.

For $n \geqq 3$, $Q_1^{\mu-1}L^{n-3}V$ is of the form (76) with $S = v$, since $\beta_3v = 0$.

Any covariant with n odd, $m = 2\mu > 0$, differs from one of rank $> m$ by a linear combination of $IQ_1^{\mu}L^n$ $(I = 1, A, \Delta)$, K^mL^n, ΔK^mL^n and, if $n > 1$, $Q_1^{\mu-1}L^{n-3}V$.

28. For $m = 2\mu > 0$, $n = 4\nu > 0$, the coefficients of $\xi_2^mx_3^n$ in

$$Q_1^{\mu}Q_2^{\mu},\quad K^mQ_2^{\nu},\quad Q_1^{\mu}F^{2\nu},\quad Q_1^{\mu}L^n,\quad K^mL^n,$$
$$Q_1^{\mu-1}Q_2^{\nu-1}G^2,\quad K^{m-2}Q_2^{\nu-1}G^2 \tag{92}$$

are respectively

$$1,\quad a_2,\quad b_3,\quad \beta_3+1,\quad a_2(b_3+1),\quad d=\beta_3(\beta_3+1),\quad a_2d.$$

These may be multiplied by any invariant. Now

$$\beta_3 + 1 + a_2 + b_3 = a_1\alpha_2,$$

$$\Delta(\beta_3 + 1) + (\Delta + A + 1)b_3 + b_3a_2 + \Delta = b_3a_1\alpha_2,$$

$$d + A + \beta_3 + \alpha_2(\Delta + b_3) = b_1(b_3 + \alpha_2) \equiv \beta,$$

$$a_2d + a_2b_3 = a_2b_1b_3 = a_2\beta,\quad Ad = Ab_1(b_3 + 1) = A\beta.$$

Hence we have a covariant (76) in which the coefficient of $\xi_2^mx_3^n$ is any linear combination of the functions (71). Hence *the*

tions of the proper degree in the x's, of the generators (77) of covariants of rank zero, I found the syzygies needed to reduce $C_1K + D_1G$ to an expression differing from the above $CK + DG$ by a covariant of rank $\geqq 2$, in which those of rank 2 are linear combinations of K^2, KG, G^2, W, Q_1 and the new one

$$V = GF^2 + \Delta Q_2 G + (\Delta + J + 1)Q_2FK$$

(89)
$$+ \Delta L^3K^2 + \Delta L^3 Q_1 = \xi_2^2 x_3^3 v + \cdots,$$

where

(90)
$$v = a_2 + b_3(1 + a_1\alpha_2).$$

The only new syzygies needed for this reduction are

$$LG \equiv Q_2L^2 + L^6 = W, \quad FLK = \Delta W + \Delta Q_1 + (J + 1)K^2,$$

(91)
$$(F^2 + L^4 + Q_2)K = (A + 1)L_3,$$

$$(\Delta + 1)(FG + KL^4 + KQ_2) + JKQ_2 = ALQ_1 + \omega L_3,$$

in which ω is an invariant not computed. Proof need not be given of these facts since we presuppose below merely the existence of relation (89) which may be verified independently. Of course, the fact that V is the only new covariant of rank 2 was a guide in the later investigation.

Covariants of Even Rank $m = 2\mu > 0$, §§ 27–29

27. First, let n be odd. In the covariant (76) replace x_3 by $x_3 + x_1$. In view of (82), we get

$$R' = f_2'\xi_2^m + f_3\xi_3^m + f_1'(\xi_1^2 + \xi_3^2)^\mu + (x_1x_2x_3 + x_1^2x_2)\phi'.$$

Using the notation (84) for f_2, we have $S_1' = S_1 + S$ in f_2'. Thus, as in § 17, S is a linear combination of the functions (74). Now $Q_1^\mu L^n$ and its products by A and $A + \Delta$ are covariants (76) with S given by (63). Using also $K^m L^n$, in which $S = a_2(b_3+1)$, and its product by Δ, we may set

$$S = k_1(b_3 + a_2) + k_2b_3a_1\alpha_2 + k_3\Delta\alpha_2 + k_4(b_3 + a_2)J.$$

In $x_1x_2x_3\phi$, let g be the coefficient of

$$x_1x_2x_3 \cdot x_2^{2\mu-1}x_3^{4\mu+n-2} = (x_2^2x_3^4)^\mu x_3^{n-1}x_1.$$

Since this must be of the form (57), we get $I = 0$, $c = e = 0$. *A covariant with* $m = 2\mu$, $n = 2$, *differs from one of rank* $> m$ *by a linear function of*

$$iMK^{m-1}, \quad K^mL^2, \quad \Delta K^mL^2, \quad GK^{m-1}, \quad IFQ_1^{\mu}, \quad L^2Q_1^{\mu}, \quad \Delta L^2Q_1^{\mu}$$
$$(i = 1,\ \Delta,\ J;\ I = 1,\ A,\ \Delta,\ J).$$

For $n > 2$, we may delete Δ from the part I of S by use of $EQ_1^{\mu}Q_2^{\nu-1}$, where E is given by (75). Without disturbing S we may delete $a_2(b_3 + 1)$ and its product by Δ from S_1 by use of $K^{2\mu+1}L^{n-3}$, since the term of $\xi_2^{2\mu}f_2$ with the coefficient S_1 is the term of highest degree in x_3 in $\xi_2^{2\mu+1}(S_1x_3^{n-3} + \cdots)$. Since $S + S_1$ is a linear combination of the functions (74),

$$\begin{aligned} S_1 = S + t_1(b_3 + a_2) + t_2a_1\alpha_2 + t_3b_3a_1\alpha_2 + t_4b_3A + t_5\Delta\alpha_2 \\ + t\Delta(b_3 + 1) + t_0(b_3 + a_2)J. \end{aligned} \tag{95}$$

Apply $(a_1a_2a_3)(b_1b_2b_3)$ to B_1, of the form (57). Hence

$$S_1 = p\rho a_1 + pa_2b_2 + pa_2\rho + ra_2 + s\rho, \quad \rho = b_1 + a_3. \tag{96}$$

Now a_1b_2 occurs in S only in the terms J, AJ of I and in the part of (95) after S only in the last term, given by (72). In these the factors of a_1b_2 are linearly independent. Hence $t_0 = 0$, $I = x(A + 1)$. The coefficient of a_1 in S_1 must vanish for $b_1 \equiv a_3$, and S_1 itself if also $a_2 \equiv 0$. Hence

$$c = t_2 = x, \quad t_1 = t_3 = t_4 = t, \quad t_5 = x + t,$$

$$\begin{aligned} S_1 = x(A + 1 + b_1b_3 + b_1\alpha_2 + a_1\alpha_2 + \Delta\alpha_2) + eAb_1(b_3 + 1) \\ + t(b_3 + a_2 + b_3a_1\alpha_2 + b_3A + b_3\Delta + a_2\Delta). \end{aligned}$$

Call ϵ the coefficient in $x_1x_2x_3\phi$ of

$$x_1x_2x_3 \cdot x_2^{2\mu}x_3^{4\mu+n-3} = (x_2^2x_3^4)^{\mu}x_1x_2x_3^{n-2}.$$

In R' of § 27, the coefficient of this product is $\epsilon + B_{n-1}$. Hence B_{n-1} is of the form (57). Interchanging the subscripts 1 and 2 in B_{n-1}, we get S_1. Thus the coefficient of a_3 in S_1 vanishes for $b_3 = a_1$. Hence $S = S_1 = 0$. *Any covariant with* $n > 2$ *differs*

covariant differs from one of rank $> m$ *by a linear function of the covariants* (92), *the products of the first three by any invariant except* 1, *the products of the fourth and fifth by* Δ *and the product of the sixth by* A.

29. For $m = 2\mu > 0$, $n = 4\nu + 2$, the coefficients of $\xi_2^m x_3^n$ in

(93) $\quad MK^{m-1}Q_2^{\nu}, \quad K^m L^n, \quad GK^{m-1}Q_2^{\nu}, \quad F^{n/2}Q_1^{\mu}, \quad L^n Q_1^{\mu},$

are respectively

$$a_2, \quad a_2(b_3 + 1), \quad a_2 b_3(b_1 + 1), \quad b_3, \quad b_3 + \alpha_1\alpha_2 + 1.$$

Linear combinations of products of these by invariants give*

$$a_2, \quad a_2\Delta, \quad a_2 J, \quad a_2 b_3, \quad \Delta a_2 b_3, \quad a_2 b_1 b_3, \quad I b_3, \quad a_1\alpha_2, \quad \Delta + b_3 a_1 \alpha_2.$$

Since S and S_1 are unaltered by the group Γ of § 15, they are linear combinations of the functions (71). Deleting the above functions $a_2, a_2\Delta, \cdots$ from S, we have

$$S = I + c\beta + eA\beta, \quad \beta = b_1(b_3 + \alpha_2),$$

where c and e are constants, and I is an invariant. Set

$$f_1 = Bx_2^n + B_1 x_2^{n-1}x_3 + \cdots + B_{n-1}x_2 x_3^{n-1} + B_n x_3^n,$$

and call σ the coefficient of

(94) $$x_1 x_2 x_3 \cdot x_2^{4\mu+n-2}x_3^{2\mu-1} = (x_2^2 x_3)^{2\mu} x_2^{n-1} x_1$$

in $x_1 x_2 x_3 \phi$. The coefficient† of (94) in R' of § 27 is $B_1 + \sigma$. Hence

$$\sigma' - \sigma = B_1,$$

if (50) replaces σ by σ'. Thus B_1 must be of the form (57).

For $n = 2$, S_2 is derived from S by applying $(a_1 a_3)(b_1 b_3)$. Then (67_2) gives S_1. Applying $(a_1 a_2)(b_1 b_2)$ to S_1, we get

$$B_1 = I + c(b_2 b_3 + b_2\alpha_1 + b_3\alpha_1) + eA(b_2 b_3 + b_2 + b_3).$$

* For the last two, use the first two of the four equations in § 28.

† The first part of R' is free of x_2, the second of x_3, while in the third part ξ_3^2 has the factor x_1^2, and in $f_1' \xi_1^{2\mu}$ there is a single term (94) and it has the coefficient B_1.

Thus

$$S = x(A + 1 + \Delta + a_1\alpha_2 + b_1b_3 + b_1\alpha_2) + k_4a_2b_1b_3 \tag{97}$$
$$+ g(A + 1 + \Delta + a_1\alpha_2)b_3 + k_5Ab_1(b_3 + 1).$$

First, let $n = 4\nu + 2$ and write $2\mu + 1$ for m. Then

$$GQ_1^{\mu}Q_2^{\nu}, \quad K^2GQ_1^{\mu-1}Q_2^{\nu}$$

have $d = \beta_3(\beta_1 + 1)$ and a_2d as the coefficients of $\xi_2^m x_3^n$. As in § 25, the coefficients of x, k_4, g, k_5 in (97) equal respectively

$$d + a_2(\Delta + b_3 + 1), \quad a_2d + a_2b_3, \quad \Delta d + a_2d + a_2J, \quad Ad.$$

The terms not containing d are combinations of the above a_2I_1 and $a_2(b_3 + 1)$ of § 23. *Any covariant with* $m = 2\mu + 1 > 1$, $n = 4\nu + 2$, *differs from one of rank* $> m$ *by a linear function of*

$$iK^mL^n, \quad I_1K^{m-1}MQ_2^{\nu}, \quad IGQ_1^{\mu}Q_2^{\nu}, \quad K^2GQ_1^{\mu-1}Q_2^{\nu}$$
$$(i = 1, \Delta;\ I_1 = 1, \Delta, J;\ I = 1, A, \Delta).$$

Next, let $n = 4\nu > 0$. In the last two covariants of the theorem below, the coefficients of $\xi_2^{2\mu+1}x_3^{4\nu}$ are $a_2b_3(b_1 + 1)$ and $\delta = b_3\beta_3(\beta_1 + 1)$. We had reached covariants in which the corresponding coefficients are a_2I and $a_2(b_3 + 1)I$. Thus we obtain the coefficient of k_4 in (97) and $\delta + \Delta a_2b_3 + a_2b_1b_3$, which equals the coefficient of g. We may therefore set $k_4 = g = 0$. Subtracting covariants of the fourth and fifth types in the theorem, we may take as S_1 the function in § 24, without disturbing S. Applying $(a_1a_2)(b_1b_2)$ to S and S_1, we get B_n and B_{n-1}. If l is the coefficient of $x_1x_2^{m+1}x_3^{2m+n-2}$ in $x_1x_2x_3\phi$, its coefficient in R' of § 30 is $l' = l + B_n + B_{n-1}$. Thus $B_n + B_{n-1}$ is of the form (57). By the coefficient of a_3b_1, $t_4 = 0$. Since the coefficient of a_3 is zero for $b_3 = a_2$, we get $x = k_5 = t_3 = 0$. Thus $S = 0$. *Any covariant with* $m = 2\mu + 1 > 1$, $n = 4\nu > 0$, *differs from one of rank* $> m$ *by a linear function of*

$$K^mL^n, \ \Delta K^mL^n, \ IK^mQ_2^{\nu}, \ iL^{n-3}Q_1^{\mu+1}, \ iL^{n-3}K^{2\mu+2}, \ G^2KQ_2^{\nu-1}Q_1^{\mu-1},$$
$$FGQ_2^{\nu-1}Q_1^{\mu} \quad (i = 1, A, \Delta;\ I = 1, \Delta, J).$$

from one of rank $> m$ *by a linear combination of*

$$iMK^{m-1}Q_2^{\nu},\quad jK^mL^n,\quad GK^{m-1}Q_2^{\nu},\quad IF^{n/2}Q_1^{\mu},\quad jL^nQ_1^{\mu},\quad EQ_1^{\mu}Q_2^{\nu-1}$$
$$(i = 1,\ \Delta,\ J;\ j = 1,\ \Delta;\ I = 1,\ A,\ \Delta,\ J).$$

COVARIANTS OF ODD RANK $m = 2\mu + 1 > 1$, §§ 30–31

30. Replacing x_3 by $x_3 + x_1$ in the covariant (76), we get

$$R' = f_2'\xi_2^m + f_3\xi_3^m + f_1'(\xi_1 + \xi_3)^m + (x_1x_2x_3 + x_1^2x_2)\phi'.$$

In $x_1x_2x_3\phi$, let g be the coefficient of $(x_1x_2^2)(x_2^2x_3)^{m-1}x_2^n$. The coefficient of the corresponding term of R' is $g' \equiv g + B$, where B is that of x_2^n in f_1. Hence B is of the form (57).

First, let n be odd. Then $S_1' = S_1 + S$ under (50), so that S is a linear combination of functions (74) with $a_2(b_3 + 1)$ and its product by Δ deleted (§ 23). Thus S is the sum of the terms (95) after the first. Applying $(a_1a_2a_3)(b_1b_2b_3)$ to B, of the form (57), we see that S is of the form (96). By these two results,

$$S = t(b_3 + a_2 + b_3a_1a_2 + b_3A + b_3\Delta + a_2\Delta).$$

If l is the coefficient of $(x_2x_3^2)^mx_3^{n-1}x_1$ in $x_1x_2x_3\phi$, that in R' is $l' = l + nB_n$. Hence, for n odd, B_n is of the form (57). Interchanging the subscripts 1, 2 in B_n, we get S. Thus the coefficient of a_3 in S vanishes for $b_3 \equiv a_1$, so that $t \equiv 0$. *Any covariant with* m *and* n *odd differs from one of rank* $> m$ *by a linear function of* K^mL^n *and* ΔK^mL^n.

31. Finally, let m be odd and n even. According as $n = 4\nu$ or $4\nu + 2$, $K^mQ_2^{\nu}$ or $K^{m-1}MQ_2^{\nu}$ is of the form (76) with a_2 as the coefficient of $\xi_2^mx_3^n$. Hence we may delete the terms a_2I_1 in (85) and hence the terms a_1I_1 in B of § 23. But (§ 30), B is of the form (57). Now a_3b_1 occurs in J and AJ of I and in b_2J of b_2I_2, having in these linearly independent multipliers. Hence

$$I = x(A + 1) + y\Delta,\quad I_2 = e + fA + g\Delta.$$

Since the coefficient of a_3 in B shall vanish for $b_3 = a_2$, and B itself if also $a_1 = 0$, we get $k_1 = x = y = k_3$, $k_2 = f = g = e$.

LECTURE V

A THEORY OF PLANE CUBIC CURVES WITH A REAL INFLEXION POINT VALID IN ORDINARY AND IN MODULAR GEOMETRY

1. *Normal Form of Cubic.*—Consider a ternary cubic form $C(x, y, z)$ with coefficients in a field F not having modulus 2 or 3. After applying a linear transformation with coefficients in F and of determinant unity, we may assume that $(1, 0, 0)$ is an inflexion point. In particular, C lacks the term x^3. If it lacks also x^2y and x^2z, its first partial derivatives vanish for $y = z = 0$. But we shall assume that the discriminant of C is not zero. Hence the coefficient of x^2 may be taken as the new variable y. At the inflexion point $(1, 0, 0)$ the tangent $y = 0$ is to be an inflexion tangent, i. e., meet the cubic in a single point. Hence C lacks the term xz^2. Thus

$$C = x^2y + 2x(\alpha y^2 + \beta yz) + \phi(y, z).$$

Replacing x by $x - \alpha y - \beta z$, we see that x^2y is now the only term involving x. If y were a factor, the discriminant would be zero. Hence the term z^3 occurs. Adding a suitable multiple of y to z, we get

$$(1) \qquad C = x^2y + gy^3 + hy^2z + \delta z^3 \qquad (\delta \neq 0).$$

2. *The Invariants s and t.*—The Hessian of (1) is

$$H = -3\delta x^2z - h^2y^3 + 9\delta gy^2z + 3\delta hyz^2.$$

The sides of an inflexion triangle form a degenerate cubic belonging to the pencil of cubics $kC + H$. The latter has the factor z only when $k = h = 0$ and the factor $y - lz$ only when $kl = 3\delta$ (as shown by the terms in x^2), where k is a root of

$$k^4 + 18\delta hk^2 + 108\delta^2gk - 27\delta^2h^2 = 0.$$

Before considering the factors involving x, we note that the

32. We have now completed the proof of the theorem:

As a fundamental system of modular covariants of the ternary quadratic form F with integral coefficients modulo 2, we may take F, its invariants A, Δ, J, its linear covariant L, its "polar" cubic covariant K, and the universal covariants Q_1, Q_2, L_3.

Incidentally, we have obtained a complete set of linearly independent covariants of each order and rank. We might then find a complete set of independent syzygies. Syzygies whose members are covariants of low rank are given in (78), (88), (91).

33. *References on Modular Geometry.*—Other aspects of the modular geometry of quadratic forms modulo 2 and, in particular, applications to theta functions have been considered by Coble.* For a treatment of non-homogeneous quadratic forms in x, y modulo p (p an odd prime), analogous to that of conics in elementary analytic geometry, but employing only real points on the modular locus, see G. Arnoux, Essai de Géométrie analytique modulaire, Paris, 1911. The earlier paper by Veblen and Bussey was cited in § 7. The paper by Mitchell was cited in § 3. Applications of modular geometries have been made by Conwell.†

The problem of coloring a map has been treated from the standpoint of modular geometry by Veblen.‡

* *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 241–276.

† *Annals of Mathematics*, ser. 2, vol. 11 (1910), pp. 60–76.

‡ *Annals of Mathematics*, ser. 2, vol. 14 (1912), pp. 86–94.

Each root of $k^3 = t$ gives an inflexion triangle

$$(6)\qquad y = \frac{3\delta}{k} z, \qquad x = \pm \tau_2 \left(y + \frac{6\delta}{k} z \right) \quad (3 \cdot 36\delta^2 \tau_2^2 = t).$$

Next, let $s \neq 0$. Each root of (3) gives an inflexion triangle

$$(7)\qquad y = \frac{3\delta}{k} z, \qquad x = \pm \sqrt{\frac{k}{3}} \left(z + \frac{k^2 - 3s}{6\delta k} y \right).$$

4. *The Parameter* δ.—If we multiply x, y, z by ρ, ρ^{-2}, ρ, we obtain from (1) a cubic with δ replaced by $\delta\rho^3$. If F is the field of all complex numbers, the field of all real numbers, or the finite field of the residues of integers modulo $3j + 2$, a prime, every element is the cube of an element of the field [in the third case, $e \equiv (e^{-j})^3$], so that the parameter δ may be taken to be unity. Although we do not use the fact below, it is in place to state here that for all further fields a new invariant is needed to distinguish the classes of cubics (1). Indeed, two cubics (1), with coefficients in F and with the same invariants s and t and discriminants not zero, are equivalent under a linear transformation with coefficients in F and having determinant unity if and only if the ratio of their δ's is the cube of an element of F.

Criteria for 9, 3 or 1 Real Inflexion Points, §§ 5–9

5. *Inflexion Points when* $s = 0$.—Let κ be a fixed root of $k^3 = t$. Let τ_1 and τ_2 be fixed roots of the equations at the end of (5) and (6). Then

$$(\tau_1/\tau_2)^2 = -3 = (1 + 2\omega)^2, \quad \omega^2 + \omega + 1 = 0.$$

Choose ω so that $\tau_1/\tau_2 = 1 + 2\omega$. Denote the lines $z = 0$, $x = \tau_1 y$, $x = -\tau_1 y$ in (5) by L_1, L_2, L_3. For each value of $i = 0, 1, 2$, denote the three lines (6) with $k = \kappa\omega^i$ by L_{1i}, L_{2i}, L_{3i}, that with the lower sign being L_{3i}. Then the 9 inflexion points and the subscripts of the 4 inflexion lines through each are given in the following table:

coefficients of this quartic equation are the values which relative invariants of a general cubic assume for the case of our cubic (1). Indeed, a linear transformation of determinant unity which replaces C by a cubic C' must replace H by the Hessian H' of C', and hence replace the inflexion triangle of C given by a root k of the quartic by that inflexion triangle of C' which is given by the same number k. We denote the invariants by*

(2) $$s = -3\delta h, \quad t = -108\delta^2 g.$$

The above quartic now becomes

(3) $$k^4 - 6sk^2 - tk - 3s^2 = 0.$$

The discriminant Δ of C is such that

(4) $$27\Delta = t^2 - 64s^3.$$

There are four distinct roots of (3) since its discriminant is $-27^3\Delta^2$.

Our earlier results are that $kC + H$ has the factor z only when $k = s = 0$ and the factor $y - 3\delta k^{-1}z$ if k is a root $\neq 0$ of (3). It has the factor $x - \tau y - \rho z$ if and only if

$$3\rho^2 = k, \quad 9\delta^2 k\tau^2 = s^2 + tk/12, \qquad k\rho^2 - 6\delta\rho\tau = s,$$

$$6\delta k\rho\tau - 9\delta^2\tau^2 - sk - t/4 = 0.$$

These conditions are satisfied if and only if k is a root of (3) and

$$\rho = k = 0, \quad 36\delta^2\tau^2 = -t \qquad (k = 0),$$

$$3\rho^2 = k, \quad 6\delta k\tau = \rho(k^2 - 3s) \qquad (k \neq 0).$$

3. *The Four Inflexion Triangles.*—First, let $s = 0$. Then $t \neq 0$ by (4). The root $k = 0$ gives the inflexion triangle with the sides

(5) $$z = 0, \quad x = \pm \tau_1 y \qquad (36\delta^2\tau_1^2 = -t).$$

* We have $s = -3^4S$, $t = -3^6T$, where S and T, given in Salmon's Higher Plane Curves, p. 189, are the invariants of the general cubic with multinomial coefficients.

$\tau_2 \equiv +1$, $\kappa \equiv -1$, we get $\omega \equiv 2$. Thus $x^2y - y^3 + 3z^3 \equiv 0$ has the 9 inflexion points $(1, 0, 0)$, $(1, 1, 0)$, $(-1, 1, 0)$, $(-2, 1, 3 \cdot 2^i)$, $(2, 1, 3 \cdot 2^i)$ $(i = 0, 1, 2)$.

6. *Inflexion Points when* $s \neq 0$, $\Delta \neq 0$.—These are $(1, 0, 0)$ and

$$(9) \qquad \left(\frac{s - k^2}{\pm 2k\sqrt{-k}}, \quad \frac{3\delta}{k}, \quad 1\right),$$

where k ranges over the roots of the quartic (3). We seek the number of real roots k for which $\sqrt{-k}$ is real. In order that the left member of (3) shall have the factors

$$(10) \qquad k^2 + wk + l, \quad k^2 - wk + m,$$

it is necessary and sufficient that

$$(11) \quad l + m - w^2 = -6s, \quad (l - m)w = t, \quad lm = -3s^2.$$

Let $t \neq 0$ (for $t = 0$ see § 9). Then $w \neq 0$ and

$$(12) \qquad 2l = w^2 - 6s + t/w, \quad 2m = w^2 - 6s - t/w.$$

Inserting these values into (11_3), we get

$$(13) \qquad w^6 - 12sw^4 + 48s^2w^2 - t^2 = 0.$$

Set $w^2 = y + 4s$. Then

$$(14) \qquad y^3 = t^2 - 64s^3 = 27\Delta.$$

For the rest of this section, let the field be that of the residues of integers modulo p, where p is an odd prime $3j + 2$. Since any integer e has a unique cube root e^{-j} modulo p, there is a single real root y of (14).

First, let $y + 4s$ be a quadratic residue of p. Then w is real and hence also l and m. The product of the discriminants of the quadratic functions (10) is seen by (11_1) and (11_3) to equal

$$(15) \qquad (w^2 - 4l)(w^2 - 4m) = -3(w^2 - 4s)^2 = -3y^2$$

and hence is a quadratic non-residue of p. Thus a single one of the quadratics (10), say the first, has a discriminant which is a

	$(1, 0, 0)$	$(\tau_2, 1, 0)$	$(-\tau_2, 1, 0)$	$\left(\tau_1, 1, \frac{\kappa\omega^i}{3\delta}\right)$	$\left(-\tau_1, 1, \frac{\kappa\omega^i}{3\delta}\right)$
	1	1	1	2	3
(8)	10	20	30	$1i$	$1i$
	11	21	31	$2, i-1$	$2, i-2$
	12	22	32	$3, i-2$	$3, i-1$

In the last two columns, i has the values 0, 1, 2; while $i - 1$ or $i - 2$ is to be replaced by the number 0, 1, 2 to which it is congruent modulo 3.

When F is the field of all real numbers, κ may be taken to be real, while just one of the numbers τ_1 and τ_2 is real. Hence 3 and only 3 of the 9 inflexion points are real. The same result is true if F is the field of the p residues of integers modulo p, where p is a prime $3j + 2 > 2$. For, κ may be taken to be integral (§ 4), while ω is imaginary and hence -3 is a quadratic non-residue of p. If $-t$ is a quadratic residue, τ_1 is real and τ_2 imaginary. If $-t$ is a non-residue, the reverse is true.

Next, let $p = 3j + 1$, so that ω is real and hence -3 a quadratic residue. By (5) and (6), τ_1 and τ_2 are both real or both imaginary according as $-t$ is a quadratic residue or non-residue of p. Hence all 9 inflexion points are real if and only if $-t$ is both a square and a cube and hence a 6th power modulo p. If $-t$ is a square but not a cube, only the first 3 inflexion points are real. If $-t$ is a quadratic non-residue, $(1, 0, 0)$ is the only real inflexion point.

A cubic with integral coefficients taken modulo p, a prime > 3, with at least one real inflexion point and with invariant $s = 0$ and invariant $t \neq 0$, has 9 real inflexion points if $p = 3j + 1$ and $-t$ is a sixth power modulo p, a single real inflexion point if $p = 3j + 1$ and $-t$ is a quadratic non-residue of p, and exactly 3 real inflexion points in all of the remaining cases.

For example, if $p = 7$ and $s = 0$, $t \neq 0$, there are 9 real inflexion points only when $t \equiv -1$. Taking $\delta \equiv 3$, $\tau_1 \equiv -2$,

are non-residues, there is a single factorization of quartic (3) into real quadratics (10) and hence certainly not four real roots. The product (15) of the discriminants of the real quadratic factors is now a quadratic residue of p. If each were a residue, there would be four real roots. Hence each is a non-residue and there is no real root. *There is a single real inflexion point if* $p = 3j + 1$, $st\Delta \neq 0$, Δ *is a cube, and if the three values of* $3\Delta^{\frac{1}{3}} + 4s$ *are not all quadratic residues of* p.

Next, let each $y_i + 4s$ be a quadratic residue of p. Then there are three ways of factoring quartic (3) into real quadratics (10). But a root common to two distinct real quadratics is real. Hence all four roots are real. The discriminant of each quadratic (10) is therefore a quadratic residue of p. Hence, by (16), l is a quadratic residue of p; similarly for the constant term of each quadratic factor. Thus the negatives of the four roots are all quadratic residues or all non-residues.

To decide between these alternatives, we need the actual roots. In $w_i^2 = y_i + 4s$, let the signs of the w_i be chosen so that

$$k^2 - w_i k + m_i = 0 \qquad (i = 1, 2, 3)$$

have a common root. As in (12),

$$2m_i = w_i^2 - 6s - t/w_i.$$

For $e \neq 1$, we find by subtraction and cancellation of $w_1 - w_e$ that

$$2k = w_1 + w_e + t/(w_1 w_e).$$

Comparing the results for $e = 2$ and $e = 3$, we get

$$w_1 w_2 w_3 = t. \tag{17}$$

Hence* the roots of (3) are

$$\begin{gathered} \tfrac{1}{2}(w_1 + w_2 + w_3), \quad \tfrac{1}{2}(w_1 - w_2 - w_3), \\ \tfrac{1}{2}(- w_1 + w_2 - w_3), \quad \tfrac{1}{2}(- w_1 - w_2 + w_3). \end{gathered} \tag{18}$$

The product of the first and $(i + 1)$th roots is seen to equal m_i

* In particular, we have deduced Euler's solution by the method of Descartes.

quadratic residue and hence has real roots. By (12_1),

$$4l(w^2-4l)w^2=-2w^6-6w^3t+36sw^4-4t^2+48stw-144s^2w^2.$$

Adding the vanishing quantity (13), we see that

$$(16)\qquad 4l(w^2-4l)w^2=-3(w^3-8sw+t)^2.$$

Since w^2-4l is a quadratic residue and -3 is a non-residue of p, it follows that l is a non-residue. Hence a single one of the roots of the first quadratic (10), and hence a single one of the roots of the quartic (3), is the negative of a quadratic residue. Thus just two of the inflexion points (9) are real.

Next, let $y+4s$ be a quadratic non-residue of p. Then there is no factorization of the quartic (3) into real quadratic factors. Nor is there a real linear factor $k-r$ and a real irreducible cubic factor. For, if so, the roots of the latter are of the form λ, λ^p, λ^{p^2} (cf. the first foot-note p. 37). Then

$$(r-\lambda)(r-\lambda^p)(r-\lambda^{p^2}),\ P=(\lambda-\lambda^p)(\lambda^p-\lambda^{p^2})(\lambda^{p^2}-\lambda)\equiv P^p \pmod{p}$$

are real, so that the discriminant of (3) is a quadratic residue. But this discriminant was seen to be $-3(81\Delta)^2$, and -3 is a non-residue. Hence (3) is irreducible modulo p. Thus $(1, 0, 0)$ is the only real inflexion point.

For $p=3j+2>2$, a cubic (1) with $st\Delta\neq 0$, has exactly three real inflexion points or a single one according as the real number $3\Delta^{\frac{1}{3}}+4s$ is a quadratic residue or non-residue of p.

7. *Cubic with $st\Delta\neq 0$, $p=3j+1$.*—Now -3 is a quadratic residue of p and there are three real cube roots $1, \omega, \omega^2$ of unity modulo p.

In this section we shall assume that Δ is a cube modulo p. Then there are three real roots y_i of (14). At least one of the y_i+4s is a quadratic residue of p since

$$\prod_{i=1}^{3}(y_i+4s)=y_1^3+64s^3=t^2.$$

If y_1+4s is a quadratic residue, while y_2+4s and y_3+4s

$\sigma = \rho l^2$, where ρ and l are integers not divisible by p. Then

$$(21)\qquad w_1 = \rho(1 + l^2), \quad w_2 = 2i\omega\rho l, \quad w_3 = i\omega^2\rho(1 - l^2).$$

We must exclude the values of l which lead to equal values of two of the w_i^2, and hence to equal y_i's, since the roots of (14) are incongruent. Now if any two of the w_i^2 in (20) are congruent, all three are congruent. But $w_1^2 \equiv w_2^2$ implies

$$1 + l^2 \equiv \pm 2i\omega l, \quad (l \mp i\omega)^2 \equiv \omega^4, \quad l \equiv \pm i\omega + e\omega^2 \qquad (e^2 \equiv 1).$$

The values $l^2 \equiv 0, \pm 1$ make one of the $w_i \equiv 0$. Hence we must exclude the 9 incongruent integral values of l:

$$(22)\qquad l = 0, \quad \pm 1, \quad \pm i, \quad \omega^2 \pm i\omega, \quad -\omega^2 \pm i\omega.$$

Using the values (21), we get

$$(23)\quad 12s = \rho^2\{(1 - \omega)(1 + l^4) - 6\omega^2 l^2\}, \quad t = 2\rho^3 l(l^4 - 1),$$

$$(24)\qquad \tfrac{1}{2}(w_1 + w_2 + w_3) = \tfrac{1}{2}\rho(1 + i\omega^2)\left(1 + \frac{i\omega l}{1 + i\omega^2}\right)^2.$$

To make the negative of the last a square, we must take

$$(25)\qquad \rho = -2(1 + i\omega^2)r^2 \qquad (r \not\equiv 0).$$

Now s, given by (23), is zero only when

$$(26)\qquad l = \omega \pm i\omega^2, \quad -\omega \pm i\omega^2.$$

The desired sets s, t are given by (23) *and* (25), *where r is any integer not divisible by p, while l is any one of the $p - 13$ positive integers $< p$ not congruent modulo p to one of the* 13 *incongruent integers* (22), (26). *The minimum p is* 37.

Second, let $p = 12q + 7$. Then $\lambda^2 \equiv -1 \pmod p$ is irreducible. Its roots i and $-i = i^p$ are Galois imaginaries. Set

$$(27)\qquad \pi = p + 1, \quad \sigma = p - 1.$$

There exists a linear function R of i with integral coefficients such that $R^{\pi\sigma} = 1$, while no lower power of R is unity. Any function of i is zero or a power of R and any integer is a power of

and hence is a quadratic residue. For given values of p, s, t, we can readily find by a table of indices the real values of the w_i and thus a real root and hence decide whether or not it (and hence each of the four roots) is the negative of a quadratic residue.

However, changing our standpoint, we shall make an explicit determination of all sets s, t for which the quartic (3) has four real roots each the negative of a quadratic residue of p.

By the definition of the w_i^2, or direct from (13),

$$(19) \qquad \Sigma w_1^2 = 12s, \quad \Sigma w_1^2 w_2^2 = 48s^2, \quad w_1^2 w_2^2 w_3^2 = t^2.$$

Let ω be a fixed integral root of $\omega^2 + \omega + 1 \equiv 0 \pmod{p}$. Then

$$0 = (12s)^2 - 3(48s^2) = \Sigma w_1^4 - \Sigma w_1^2 w_2^2$$
$$= (w_1^2 + \omega w_2^2 + \omega^2 w_3^2)(w_1^2 + \omega^2 w_2^2 + \omega w_3^2).$$

Interchanging w_2 and w_3, if necessary, we have

$$(20) \qquad w_1^2 + \omega w_2^2 + \omega^2 w_3^2 \equiv 0 \qquad \pmod{p}.$$

Conversely, if the w_i^2 are any quadratic residues satisfying (20) and if we define s and t by (19_1) and (17), we obtain a quartic (3) with the four real roots (18). If we permute w_1, w_2, w_3 cyclically we obtain solutions of (20) leading to the same s and t and to the same four roots (18).

Our first problem is therefore to find all sets of solutions of (20). To this end it is necessary to treat separately the cases -1 a quadratic residue and -1 a non-residue; viz., $p = 12q + 1$ and $p = 12q + 7$ (since already $p = 3j + 1$).

First, let $p = 12q + 1$. Then $-1 \equiv i^2 \pmod{p}$, where i is an integer. Set

$$2\rho = w_1 - i\omega w_3, \quad 2\sigma = w_1 + i\omega w_3.$$

Then (20) becomes

$$4\rho\sigma = -\omega w_2^2 = (i\omega^2 w_2)^2,$$

so that $\rho\sigma$ must be a quadratic residue. Hence we may take

But

$$(31)\qquad (\omega + i\omega^2)(\omega - i\omega^2) = -1, \quad \omega + i\omega^2 = R^{v\sigma/2}, \quad \omega - i\omega^2 = -R^{-v\sigma/2} \qquad (v \text{ odd}).$$

Hence we must exclude the four cases in which

$$(32)\qquad \eta \equiv j + \tfrac{1}{2}(\pm v + 1), \quad j + \tfrac{1}{2}(\pm v + \pi + 1) \qquad (\mathrm{mod}\ \pi),$$

these four values being incongruent.

No one of the w's in (29) is zero, since e is odd by (28), so that $e \not\equiv 0, \pi/2 \pmod{\pi}$. By ($19_1$) and (17),

$$(33)\qquad 48s = (1 - \omega)(R^{2e} + R^{2pe}) + 6\omega^2 R^{2\pi\eta}, \quad 4t = -iR^{\pi\eta}(R^{2e} - R^{2pe}).$$

Finally, we must here exclude the cases in which $s = 0$. Combining $\Sigma w_1^2 = 0$ with (20), we obtain the necessary and sufficient condition $w_1^2 = \omega w_3^2$ for $s = 0$. But $w_1 = \pm \omega^2 w_3$, in connection with (29), gives

$$R^e(1 \pm i\omega) = R^{pe}(-1 \pm i\omega), \quad R^e(\omega \pm i\omega^2)^2 = R^{pe}.$$

Thus, by (31), the condition is that $e \pm v\sigma \equiv pe \pmod{\pi\sigma}$ or $e \equiv \pm v \pmod{\pi}$. Then, by (28), η is congruent modulo π to one of the values (32) decreased by $\pi/4$. *Hence the desired sets s, t are given by (33), subject to (28), in which the 8 incongruent η's given by (32) and those values decreased by $\pi/4$ are excluded. In particular, $p > 7$.*

For $p = 19$, the only admissible pairs are

$$s = 2 \cdot 2^{2l}, \quad t = 6(-2)^{3l} \quad (l = 0, 1, \cdots, 8).$$

For any l, the negatives of the roots of quartic (3) are the products of $-3 \equiv 4^2, 4, 7 \equiv 8^2, -8 \equiv 7^2$ by $(-2)^l$ and hence are quadratic residues of 19 since $-2 \equiv 6^2$.

For $p = 31$, the only pairs are

$$s = 3^{2l}, \quad t = 5(-3)^{3l}; \quad s = -3^{2l}, \quad t = 13(-3)^{3l} \quad (l = 0, \cdots, 15),$$

the negatives of the roots of (3) being the products of $7, -11, -12, -15$ and $-3, 5, 9, -11$, respectively, by $(-3)^l$, and hence are quadratic residues of 31.

R^π, a primitive root of p. Hence we may set

$$\omega^2 w_2 = R^{\pi\eta}, \quad w_1 + \omega w_3 i = R^e, \quad w_1 - \omega w_3 i = R^{pe},$$

where $0 \leqq \eta < \sigma$, $0 \leqq e < \pi\sigma$. Then (20) is equivalent to

$$R^{\pi e} + R^{2\pi\eta} = 0, \quad \pi e \equiv 2\pi\eta + \tfrac{1}{2}\pi\sigma \qquad (\mathrm{mod}\ \pi\sigma).$$

The last condition is equivalent to

$$e = 2\eta + \sigma/2 + j\sigma \qquad (0 \leqq j < \pi). \tag{28}$$

We have

$$w_2 = \omega R^{\pi\eta}, \quad 2w_1 = R^e + R^{pe}, \quad 2w_3 = -i\omega^2(R^e - R^{pe}),$$

$$2\omega^2 \Sigma w_1 = 2R^{\pi\eta} + (\omega^2 - i\omega)R^e + (\omega^2 + i\omega)R^{pe},$$

$$\begin{aligned}&(\omega^2 - i\omega)(\omega^2 + i\omega) = -1,\\ &(\omega^2 - i\omega)^\pi = -1, \quad \omega^2 - i\omega = R^{f\sigma/2} \qquad (f \text{ odd}),\end{aligned} \tag{29}$$

$$\begin{aligned}2\omega^2 \Sigma w_1 &= 2R^{\pi\eta} + R^{e+f\sigma/2} - R^{pe-f\sigma/2}\\ &= R^{\pi[j-(f+1)/2]}(R^{\eta-j+p(f+1)/2} + R^{p\eta-pj+(f+1)/2})^2.\end{aligned}$$

The last binomial is its own pth power and hence is real. We desire that the root $\frac{1}{2}\Sigma w_1$ shall be the negative of a quadratic residue and hence a non-residue. Since R^π is a primitive root of p, the condition is that $j - (f+1)/2$ shall be odd:

$$f = 2l - 1, \quad j - l = \text{odd}. \tag{30}$$

We must exclude the values making $w_1^2 = w_2^2$:

$$0 = 2R^{\sigma/2}(w_1 \mp w_2) = R^{2\eta+\sigma+j\sigma} \mp 2\omega R^{\pi\eta+\sigma/2} - R^{2p\eta-j\sigma},$$

the second term having been simplified by use of

$$R^{\pi\sigma/2} = -1, \quad R^{p\sigma} = R^{-\sigma}.$$

Completing the square of the first two terms, we get

$$(R^{\eta+\sigma(j+1)/2} \mp \omega R^{p\eta-\sigma j/2})^2 = (\omega^2 + 1)R^{2p\eta-\sigma j}.$$

Now $\omega^2 + 1 = -\omega = (ci\omega^2)^2$, where $c = 1$ or -1. Hence

$$R^{\eta+\sigma(j+1)/2} = (\pm\omega + ci\omega^2)R^{p\eta-\sigma j/2}.$$

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8. *Case* $p = 3j + 1$, $st\Delta \neq 0$, Δ *not a Cube.*—The roots of (14) are now Galois imaginaries y, y^p, y^{p^2}. As at the beginning of § 7,

$$t^2 = (y + 4s)(y^p + 4s)(y^{p^2} + 4s) \equiv (y + 4s)^{1+p+p^2}.$$

Raise each member to the power $(p - 1)/2$. We see that $y + 4s$ is the square of an element, say w, of the Galois field of order p^3. The first root (18) is $\frac{1}{2}(w + w^p + w^{p^2})$ and equals its own pth power, and hence is real. This is not true of the remaining roots (18), since $w^p \neq w$, or since a real quadratic factor would imply that w is real. Hence *the quartic has a single real root.*

For $p = 7$, the only cases in which the negative of the single real root is a quadratic residue are $t = -1$ or 3, $s = -1, -2, 3$; $t = 2$, s arbitrary $\neq 0$. For $p = 13$, the only cases are

$$\pm t = 4, 5, 6; \quad s = -1, -3, 4 \qquad (s^3 \equiv -1);$$

$$\pm t = 1, 5, 6; \quad s = -2, -5, -6 \qquad (s^3 \equiv 5);$$

and $\pm t = 3$, $-s$ equals one of the preceding six values of s.

9. *Cubic with* $t = 0$, $s \neq 0$.—In this case, (3) becomes

$$(k^2 - 3s)^2 = 12s^2.$$

If there be a real root k, 3 is a quadratic residue of p, and

$$k^2 = ls, \quad l = 3 \pm 2\sqrt{3}.$$

First, let $p = 3j + 2$, so that -3 is a quadratic non-residue of p. Then -1 must be a non-residue of p and hence $p = 12r + 11$. The product of the two l's is -3, so that a single value of k^2 is a quadratic residue. Since the two real k's are of opposite sign, there is a single real root k whose negative is a quadratic residue. *For* $t = 0$, $s \neq 0$, *and* $p = 12r + 5$, *there is a single real inflexion point; for* $p = 12r + 11$, *there are just three real inflexion points.*

Finally, let $p = 3j + 1$, so that -3 is a quadratic residue of p. If $p = 12r + 7$, then 3 is a non-residue, and there is no real k and hence a single real inflexion point. If $p = 12r + 1$, the four roots k are all real or all imaginary. For $p = 13$, $k^2 \equiv -2s$ or $-5s$, and $-k$ is a quadratic residue if and only if $k^6 \equiv 1$, $s^3 \equiv 8$, $s \equiv 2, 5, 6$. For $p = 37$, $k^2 \equiv -4s$ or $10s$, and $-k$ is a residue if and only if $s^9 \equiv 1$.

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LECTURE I

A GENERAL SURVEY OF THE FIELD

§ 1. Analytic Functions of Several Complex Variables

In the decades which lay between Cauchy's prime and the beginnings of the modern French school, the theory of functions of a single complex variable made rapid progress, the chief advances taking place on German soil. Simultaneously with these developments, important problems in the theory of analytic functions of several complex variables were attacked and the theorems connected with them divined with an insight worthy of the genius of a Riemann and a Weierstrass.

The elementary functions of several real variables admit extension into the complex domain and are seen to be developable there by Taylor's theorem, — a result to which the elementary theory of infinite series and an obvious extension of Cauchy's integral formula alike lead.

It was natural, then, to define a function of several complex variables generally with Weierstrass as one which can be developed by Taylor's theorem in the neighborhood of any ordinary point of its domain of definition; or, following Cauchy, as one which is analytic in each variable separately and continuous in all taken at once.*

* Cauchy, Turin memoir, 1831, = Exercices d'analyse, 2 (1841), p. 55; Jordan, Cours d'analyse, 1, 2d ed., 1893, § 206. The condition of continuity is introduced to simplify the proofs. It is a consequence of the former condition;

By the aid of this theorem he proved the extension of Riemann's theorem relating to removable singularities,* at least for the case that the given function can be expressed, in that part of the neighborhood of the given point where it is defined, as the quotient of two functions each analytic at the point.

It would be of interest to know whether Weierstrass ever considered the theorem in its general form. I recall no passage in his writings which contains such a reference. Is it possible that the restricted form just mentioned was sufficient for all the applications of this important theorem which he met?

§ 2. Jacobi's Theorem of Inversion and the Abelian Functions

Toward the close of the eighteenth century the way was paved, through Legendre's researches in the theory of the elliptic integrals, for some of the most important advances which have been made in analysis since the invention of the calculus,—those which cluster about the elliptic functions and their generalizations, the Abelian and the automorphic functions. Jacobi, following a line of thought which Abel had initiated, was led to formulate the problem of inversion which bears his name.†

The first solutions of this problem which appeared, restricted to the case $p = 2$,—those of Göpel (1847) and Rosenhain (1846–51),—were based on the theta functions of two arguments.‡ Weierstrass§ and Riemann|| arrived independently at solutions in the general case of the Abelian integrals corresponding to an

* Cf. III, § 4.

† Jacobi, Considerationes generales de transcendentibus Abelianis, 1832; Ges. Werke, 2, p. 5. For a statement of the general problem cf. Neumann, Abelsche Integrale, 2d ed., 1884, Chs. 14, 15; Appell et Goursat, Fonctions algébriques, Ch. 10. For an account of the history of this problem cf. Krazer's Festrede: Zur Geschichte des Umkehrproblems der Integrale, Karlsruhe, 1908.

‡ Jacobi and Göpel independently extended the elliptic thetas to the thetas of several arguments; cf. Krazer, l. c., pp. 17, 18.

§ Beitrag zur Theorie der Abelschen Integrale, Braunsberg, 1849, = Werke, 1, p. 111; *Journ. für Math.*, 47 (1854) p. 289 = Werke, 1, p. 133; ibid., 52 (1856), p. 285 = Werke, 1, p. 297. Also Werke, 4.

|| *Journ. für Math.*, 54 (1857), pp. 101/155 = Werke, 1 ed., p. 81; 2d ed., p. 88.

The Factorial Function and Analytic Continuation. One of the problems with which mathematicians had occupied themselves without obtaining satisfactory results was that of extending the definition of the function $n!$ to a continuous range of values for the argument. This question Weierstrass* took up, examining the work of his predecessors and showing that a satisfactory solution could be reached on the basis of the principle of analytic continuation, the functions considered being dependent on several variables. Thus these functions contributed at that early time to the recognition of the importance of the conception of the monogenic analytic configuration.

Existence Theorems. Cauchy had established the first existence theorems for ordinary differential equations and implicit functions.† In his further study of these problems he developed the method of power series and *séries majorantes*.‡

The extension to the case of partial differential equations was direct, and the results thus obtained were of importance. For, while much of the theory of these equations appeared plausible from geometric considerations of a somewhat crude sort or from analogy with special examples yielding an explicit solution, a secure foundation had hitherto been lacking.

Weierstrass's Theorem of Factorization. If a mathematical theory is to gain its independence and take its place among the powers, it must recognize its own peculiar problems and obtain methods for dealing with them. One of the earliest distinctive theorems which became known in the theory of functions of several complex variables is the theorem of factorization, due to Weierstrass.§

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cf. below, Lecture II, § 5. Such citations will be made in the following pages as II, § 5.

In order not to interrupt the course of the general account with which we are now engaged, the consideration of a number of detailed consequences which follow from the definition will be postponed to a later paragraph; cf. II, §§ 1, 2.

* *Journ. für Math.*, 51 (1856), p. 1; Werke, 1, p. 153.

† Cf. Enzyklopädie der math. Wiss., II *B* 1, p. 103, and ibid. II *A* 4*a*, p. 201.

‡ Turin memoir, 1831; Exercices d'analyse, 1 (1840), p. 327.

§ Cf. IV, § 1. The theorem dates from 1860.

which every period (P) can be expressed linearly with integral coefficients:

$$(P) = m'(P') + m''(P'') + \cdots + m^{(k)}(P^{(k)}).$$

Such a set of periods is called a *primitive scheme*, or *set*, of periods.*

A periodic function which is a constant or which depends on fewer than n arguments will evidently not come under this definition. This will also be the case if, on making a suitable non-singular linear transformation of the arguments, $f(z_1, \cdots, z_n)$ goes over into a function of fewer than n arguments. All other periodic functions do come under this definition, the functions excluded being precisely those which admit infinitely small periods.

It is a theorem due to Riemann† that a k-fold periodic function of p-independent variables cannot exist‡ when $k > 2p$. On the other hand, the Abelian functions have led to $2p$-fold periodic functions of p complex arguments, and such functions can also be formed by means of quotients of theta functions of p arguments.

Theta Functions with Several Arguments.—The fundamental theta function of a single argument§ can be defined by a series as follows:

$$\vartheta(u) = \vartheta(u, a) = C \sum_{n=-\infty}^{\infty} e^{an^2+2nu}, \qquad C \neq 0,$$

where

$$a = r + si$$

and

$$r = \Re(a) < 0.$$

* I avoid the term primitive *system* of periods because of the confusion which would thus be introduced, due to the other sense, above mentioned, in which the words *system of periods* are used.

† *Journ. für Math.*, 71 (1859), p. 197 = Werke, 1 ed., p. 276; 2d ed., p. 294. Cf. also Weierstrass, *Berliner Monatsber.*, 1876, p. 680 = Werke, 2, p. 55.

‡ The maximum number of periods which an integral function can have is p. Hermite, in Lacroix's Calcul différentiel et calcul intégral, vol. 2, 6th ed., 1862, p. 390.

§ This function appears in Fourier's Théorie analytique de la chaleur, 1822, p. 333. It is usually thought of as due to Jacobi, who was the first to recognize its importance in the theory of the elliptic functions; Fundamenta nova, 1829, = Werke, 1, p. 228.

arbitrary algebraic configuration. In these investigations both mathematicians were led to the study of the theta functions of p arguments,—in fact, Weierstrass, to whom the generalized thetas were at that time unknown, thus came to discover the form of these functions.*

The Abelian functions themselves are not single-valued. They are the roots of algebraic equations of degree p, whose coefficients are single-valued functions having only non-essential singularities in the finite region of the space of their p complex arguments and admitting $2p$ independent periods; cf. § 3.

Here, then, is a general class of functions of several variables, to which Jacobi's problem of inversion has directly led,—the class which corresponds to the doubly periodic functions of a single variable.

§ 3. Periodic Functions

To state more precisely what is meant by periodicity, it is this. The function $f(z_1, \cdots, z_n)$ is said to admit the period† $(P) = (P_1, \cdots, P_n)$ if

$$f(z_1 + P_1, z_2 + P_2, \cdots, z_n + P_n) = f(z_1, \cdots, z_n),$$

where $P_1, \cdots, P_n$ are constants.

We shall restrict ourselves here, unless the contrary is explicitly stated, to functions which are single-valued and have no other than non-essential singularities (III, § 2) in the finite region of space.

If (P) and (Q) are two periods, then $(P) + (Q) = (P_1 + Q_1, \cdots, P_n + Q_n)$ is evidently also a period. Moreover, $(-P) = (-P_1, \cdots, -P_n)$ is a period.

A function $f(z_1, \cdots, z_n)$ is said to be k-fold periodic if there exist k periods (P'), (P''), $\cdots$ $(P^{(k)})$, and no fewer, in terms of

* For their definition cf. § 3.

† Weierstrass uses the term *system of periods* (Periodensystem), i. e., *simultaneous system of periods*, to denote this complex, which may be thought of as a vector in space of $2p$ dimensions. The briefer term *period* would seem to suffice.

from the series,* which we write at length for the typical case $p = 3$.

$$\vartheta(u_1 + \pi i, u_2, u_3) = \vartheta(u_1, u_2, u_3),$$

$$\vartheta(u_1, u_2 + \pi i, u_3) = \vartheta(u_1, u_2, u_3),$$

$$\vartheta(u_1, u_2, u_3 + \pi i) = \vartheta(u_1, u_2, u_3),$$

$$\vartheta(u_1 + a_{11}, u_2 + a_{21}, u_3 + a_{31}) = e^{-2u_1 - a_{11}}\vartheta(u_1, u_2, u_3),$$

$$\vartheta(u_1 + a_{12}, u_2 + a_{22}, u_3 + a_{32}) = e^{-2u_2 - a_{22}}\vartheta(u_1, u_2, u_3),$$

$$\vartheta(u_1 + a_{13}, u_2 + a_{23}, u_3 + a_{33}) = e^{-2u_3 - a_{33}}\vartheta(u_1, u_2, u_3).$$

The vectors in $2p$-dimensional space corresponding to the $2p$ columns in the array

$$(1)\qquad \begin{array}{ccc|ccc} \pi i & 0 & 0 & a_{11} & a_{12} & a_{13} \\ 0 & \pi i & 0 & a_{21} & a_{22} & a_{23} \\ 0 & 0 & \pi i & a_{31} & a_{32} & a_{33} \end{array}$$

form the edges of a true prismatoid, F, and a periodic function corresponding to F can be formed as follows. Let $2pn = 6n$ complex numbers α_{kl}, β_{kl}, $k = 1, \cdots, p = 3$, $l = 1, \cdots, n$, be so chosen that

$$\alpha_{11} + \cdots + \alpha_{1n} = \beta_{11} + \cdots + \beta_{1n},$$

$$\alpha_{21} + \cdots + \alpha_{2n} = \beta_{21} + \cdots + \beta_{2n},$$

$$\alpha_{31} + \cdots + \alpha_{3n} = \beta_{31} + \cdots + \beta_{3n},$$

but that these numbers are otherwise non-specialized. Then the quotient

$$\frac{\vartheta(u_1+\alpha_{11}, u_2+\alpha_{21}, u_3+\alpha_{31}) \cdots \vartheta(u_1+\alpha_{1n}, u_2+\alpha_{2n}, u_3+\alpha_{3n})}{\vartheta(u_1+\beta_{11}, u_2+\beta_{21}, u_3+\beta_{31}) \cdots \vartheta(u_1+\beta_{1n}, u_2+\beta_{2n}, u_3+\beta_{3n})}$$

will represent a function admitting as a primitive scheme of periods the above scheme (1). It is sufficient to take $n = 2$.

As regards the proof of this theorem, it is clear that the above quotient admits each period of the scheme (1); but it is not

* Cf. Krazer, Lehrbuch der Thetafunktionen, Chap. 1.

It has the properties:

$$\vartheta(u + \pi i) = \vartheta(u),$$

$$\vartheta(u + a) = e^{-2u-a}\vartheta(u);$$

and it has, moreover, a single root of the first order in the parallelogram F, two sides of which are the vectors $(0, \pi i)$ and $(0, a)$.

By means of this function, doubly periodic functions can be formed as follows. Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ be any $2n$ points so chosen that

$$\sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k,$$

and that, furthermore, the points of the parallelogram F that are congruent to them are distinct. Then the quotient

$$\frac{\vartheta(u + \alpha_1) \cdots \vartheta(u + \alpha_n)}{\vartheta(u + \beta_1) \cdots \vartheta(u + \beta_n)}$$

will evidently represent a doubly periodic function with the periods πi and a.

The fundamental theta function of p arguments is given by the following series:

$$\vartheta(u_1, \cdots, u_p) = C \sum e^{\Gamma + 2\sum\limits_{k=1}^{p} n_k u_k}, \qquad C \neq 0,$$

where

$$\Gamma = \Gamma(n_1, \cdots, n_p) = \sum_{k,\, l=1}^{p} a_{kl} n_k n_l, \qquad a_{kl} = a_{lk},$$

$$a_{kl} = r_{kl} + i s_{kl},$$

and the real part of $\Gamma(x_1, \cdots, x_p)$, where $x_1, \cdots, x_p$ denote real variables, namely

$$\sum_{k,\, l=1}^{p} r_{kl} x_k x_l,$$

is a definite negative quadratic form.

The function has the following properties, readily deducible

parallelogram of periods for $p = 1$ and which forms a fundamental region for the function, — the prismatoid, F, — depends, after reduction to normal form, as we shall presently see, on p^2 complex, or $2p^2$ real constants.

With reference to this normal form, let $z_1, \cdots, z_p$ be the original arguments and let the original $2p$ periods, which are linearly independent, be written in the columns of the following array:

$$(2)\qquad \begin{array}{c|cccc|cccc} z_1 & \omega_{11} & \cdots & & \omega_{1p} & \omega'_{11} & \cdots & & \omega'_{1p} \\ z_2 & \omega_{21} & \cdots & & \omega_{2p} & \omega'_{21} & \cdots & & \omega'_{2p} \\ \vdots & . & . & . & . & . & . & . & . \\ z_p & \omega_{p1} & \cdots & & \omega_{pp} & \omega'_{p1} & \cdots & & \omega'_{pp} \end{array}.$$

Then at least one of the p-rowed determinants taken from the matrix of the $2p^2$ ω's corresponding to the scheme (2) will be different from 0.* Let this be the determinant $\pm \Sigma \omega_{11} \cdots \omega_{pp}$. If now we set

$$\pi i z_k = \omega_{k1} u_1 + \cdots + \omega_{kp} u_p,$$

the scheme of periods for the transformed function $f(z_1, \cdots, z_p) = F(u_1, \cdots, u_p)$ will be as follows:

$$\begin{array}{c|cccc|ccc} u_1 & \pi i, & 0, & \cdots, & 0 & a_{11}, & \cdots, & a_{1p} \\ u_2 & 0, & \pi i, & \cdots, & 0 & a_{21}, & \cdots, & a_{2p} \\ \vdots & . & . & . & . & . & . & . \\ u_p & 0, & 0, & \cdots, & \pi i & a_{p1}, & \cdots, & a_{pp} \end{array}.$$

Thus we have in the a_{kl}'s p^2 complex, or $2p^2$ real constants. If these be given non-specialized values, we are led to a true $2p$-dimensional prismatoid.

To any n-dimensional prismatoid F correspond real analytic functions of n real variables with n periods, for which F is a fundamental domain. If, then, in the case before us, the most

* This theorem is contained in a paper by Kronecker, *Berliner Sitzungsber.*, 29, (1884), p. 1071. Its proof follows, however, readily on developing systematically the elements of the periodic functions of several complex variables from a geometric standpoint.

clear that the α's and β's can be so chosen that this scheme is primitive for the function. This is, however, the case.*

A second mode of obtaining $2p$-fold periodic functions belonging to the scheme (1) is as follows. The functions

$$\frac{\partial^2 \log \vartheta}{\partial z_k \, \partial z_l}, \qquad k, l = 1, \cdots, p,$$

obviously admit the periods of (1), and it is readily shown that they admit only such periods as are expressible linearly with integral coefficients in terms of these.† And now it can be proven that a linear combination of the above functions can be so chosen as to yield a function belonging to the scheme (1). This statement is made by Wirtinger, l. c., but the proof is far from obvious.‡

The number of essential constants on which an algebraic configuration of deficiency $p > 1$ depends is $3p - 3$, — the so-called *moduli*. For $p = 2$ and $p = 3$ this number is the same as the number of complex constants in the theta function, namely $\frac{1}{2}p(p + 1)$. But for $p > 3$ the latter number is larger, and hence the Abelian functions of p arguments, — or rather the symmetric functions of their multiple determinations, — are not the most general $2p$-fold periodic functions.

§ 4. The Theta Theorem.

Can all $2p$-fold periodic functions with only non-essential singularities in the finite region be expressed in terms of theta functions of p arguments? The answer to this question is affirmative, and is the noted theta theorem due to Riemann and Weierstrass.

At first sight a mere count of constants appears to discredit the theorem. For the general theta function of p arguments depends on but $\frac{1}{2}p(p + 1)$ complex, or $p(p + 1)$ real constants, namely, the a_{kl} subject to the equations $a_{kl} = a_{lk}$, while the region of $2p$-dimensional space which is the analogue of the

* Cf. a forthcoming paper by the author. (Note of December 29, 1913.)

† Wirtinger, *Monatshefte f. Math. u. Phys.*, 6 (1895), p. 96, § 16.

‡ Cf. a forthcoming paper by the author. (Note of January 18, 1914.)

denominator vanish, will be in reduced form.* It is this theorem, too, on which Appell's proof cited above rests.

All of these proofs involve a considerable amount of analytical developments. Weierstrass was led, in the course of his analysis, — and it may be remarked in passing that he edited his proof with minute care, — to emphasize the importance of an accurate definition of the monogenic analytic configuration of the mth grade (Stufe) in the domain of n complex variables. He points out that it will not do to start with the points for which certain of the coordinates chosen as dependent variables are analytic in the remaining coordinates considered as independent variables, and then adjoin all limiting points to the set thus obtained. For, in the case of two variables, he says, it may happen that one would thus obtain all the points of space.†

Furthermore, in the proof as Weierstrass originally conceived it, — the final proof which appeared in his collected works is modified in essential respects, — two general theorems relating to periodic functions play an essential rôle. They are these.‡

I. Any $2p$-fold periodic function of p variables is an algebraic function of p independent $2p$-fold periodic functions belonging to the same prismatoid. Or, otherwise expressed:

Between any $p + 1$ $2p$-fold periodic functions of p variables there exists an algebraic relation.

II. Any $2p$-fold periodic function of p variables is expressible rationally in terms of $p + 1$ suitably chosen $2p$-fold periodic functions belonging to the same prismatoid.

These theorems have been generalized by Picard and Wirtinger for automorphic functions of several variables; cf. §§ 5, 6.

Poincaré's potential functions undoubtedly form a powerful instrument of analysis in dealing with the singularities of func-

* Cf. IV, § 1.

† Werke, 3, p. 96.

‡ *Berliner Monatsberichte*, 1869, p. 855 = Werke, 2, p. 46. These theorems have been treated by Poincaré, *C. R.*, 124 (1897), p. 1407; Wirtinger, *Sitzungsber. der Wiener Akad.*, 108 (1899), p. 1239; and Blumenthal, *Math. Ann.*, 58 (1904), p. 497; cf. also *Math. Ann.*, 56 (1903), pp. 510, 512.

general $2p$-fold periodic analytic functions of p complex variables are to be represented by means of quotients of thetas with p arguments, this means that the prismatoid is here subject to essential restrictions, since $p(p+1) < 2p^2$.

That this is, in fact, the case was discovered independently by Riemann and Weierstrass, and thus the first step was taken toward the establishment of the theta theorem.

Riemann never published a proof of the theorem. He communicated his results to Hermite* in 1860. Weierstrass's proof was not given in detail till the appearance of his collected works,† though he had published a number of notes bearing on a proof, and had stated the theorem in a letter to Borchardt.‡

In the early eighties Poincaré and Picard§ constructed proofs of the theorem, which, it turned out, were essentially the same as Weierstrass's. Appell‖ gave a proof in 1891 along different lines. Then came a proof by Wirtinger,¶ which has much in common with Weierstrass's proof, at that time unpublished.

Shortly after, Poincaré** gave a new proof, in which the method is that of potential functions in hyperspace. Kronecker†† had already surmized that this method would lead to fruitful results in the theory of functions of several complex variables.‡‡ Poincaré had used this method in an earlier paper, in proving the theorem that a function of two complex variables which has no other than non-essential singularities in finite space, can be expressed as the quotient of two integral functions, and that this quotient, moreover, at any point at which both numerator and

* Cf. Lacroix, Calcul différentiel et calcul intégral, vol. 2, 6th ed., 1862, p. 390.

† Werke, 3, 1903, p. 53.

‡ *Journ. für Math.*, 89 (1880), p. 8 = Werke 2, p. 133. *Berliner Monatsberichte*, 1869, p. 855 = Werke 1, p. 46.

§ *C. R.*, 97 (1883), p. 1284. Poincaré, *Acta*, 22 (1898), p. 90.

‖ *C. R.*, 110 (1890), pp. 32, 181; *Journ. de Math.* (4), 7 (1891), p. 157.

¶ *Monatshefte f. Math. u. Phys.*, 6 (1895), p. 69. Cf. also ibid., 7 (1896), p. 1.

** *Acta*, 22 (1898), p. 89. Cf. also ibid., 26 (1902), p. 43.

†† *Berliner Monatsberichte*, 1869, pp. 159, 688 = Werke, 1, p. 198.

‡‡ Cf. two memoirs by Baker, *Transactions Cambridge Phil. Soc.*, 18 (1900), p. 408, and *Proceedings London Math. Soc.* 2), 1 (1904), p. 14.

$$(2)\qquad \begin{cases} \left(x, y \;\middle|\; \dfrac{ax+b}{cx+d}, \quad \dfrac{\alpha y+\beta}{\gamma y+\delta}\right), \\ \left(x, y \;\middle|\; \dfrac{a'y+b'}{c'y+d'}, \quad \dfrac{\alpha' x+\beta'}{\gamma' x+\delta'}\right). \end{cases}$$

Hypermodular Functions.—The first papers to appear in this field dealt with groups of the type (1). Picard* began by investigating a class of functions of two independent variables analogous to the elliptic modular functions. It is a familiar fact that a hypergeometric integral

$$\int_g^h \frac{dt}{\sqrt{t(t-1)(t-x)}},$$

where g, h denote any two of the four points 0, 1, ∞, x, is a solution of the linear differential equation

$$(x^2 - x)\frac{d^2y}{dx^2} + (1-2x)\frac{dy}{dx} + \tfrac{1}{4}y = 0.$$

Let ω_1, ω_2 be two linearly independent solutions of this equation, and set

$$\frac{\omega_2}{\omega_1} = f(x).$$

Then the equation

$$f(x) = u$$

defines x as a function of u, and this function is analytic throughout the whole upper half of the u-plane, but cannot be continued analytically beyond this region.

Picard passes to analogous functions of x, y, namely those defined by one of the integrals

$$\int_g^h \frac{dt}{\sqrt[3]{t(t-1)(t-x)(t-y)}},$$

where g, h denote any two of the five points, 0, 1, ∞, x, y. These functions satisfy a simultaneous system of linear partial differ-

* *C. R.*, 93 (1881), p. 835; ibid., 94 (1882), p. 579; *Acta*, 2 (1883), p. 114. Alezais, "Sur une classe de fonctions hyperfuchsiennes," etc., Paris, 1901.

tions of several complex variables. He carries his proof through only to the point of showing that the given function can be written as the quotient of two Jacobian functions. The latter functions are defined as follows.

Jacobian Functions. Let $\omega_{\alpha\beta}$, $\alpha = 1, \cdots, p$; $\beta = 1, \cdots, 2p$, be a primitive scheme of periods, and let $f(z_1, \cdots, z_n)$ be an integral function of its p arguments. If, for every period from this scheme, a relation of the form holds:

$$f(z_1 + \omega_{1\beta}, \cdots, z_p + \omega_{p\beta}) = e^{L_\beta(z)} f(z_1, \cdots, z_p),$$

where $L_\beta(z)$ is a linear (homogeneous or non-homogeneous, but integral) function of $z_1, \cdots, z_p$, then f is called a Jacobian function.

The Jacobian functions have been studied at length in two memoirs by Frobenius,* and in a paper by Wirtinger.† A Jacobian function can be expressed in terms of theta functions of p arguments.

§ 5. Automorphic Functions of Several Variables

The brilliant results obtained by Klein and Poincaré in the early eighties, in their researches relating to the automorphic functions of a single complex variable turned the attention of mathematicians towards functions of several complex variables which admit a discrete group of linear transformations into themselves, and we find from that time to the present day a steady stream of papers in this field.

Here, however, at the very threshold of the subject, two types of groups present themselves, corresponding on the one hand formally to the linear transformations of projective space:

$$(1) \qquad \left(x, y \;\middle|\; \frac{a'x + b'y + c'}{ax + by + c}, \quad \frac{a''x + b''y + c''}{ax + by + c}\right),$$

and on the other, to those of the space of analysis:

* *Journ. für Math.*, 97 (1884), p. 16 and p. 188.

† *Monatshefte für Math. u. Phys.*, 7 (1896), p. 1.

mentioned at the outset, Riemann's researches on binary families. In fact, Appell* had just been engaged in extending these results to quaternary families of functions of two independent variables, and Picard† had himself been working in the same field.

§ 6. Continuation. Hyperfuchsian and Hyperabelian Functions

A further paper of Picard‡ deals with functions $F(u, v)$ meromorphic in their domain of definition, D, which consists of the interior of the hypersphere

$$u'^2 + u''^2 + v'^2 + v''^2 < 1,$$

and admitting a group of transformations into themselves of type (1). The fundamental domain of the group lies wholly within D. There is an allied system of simultaneous linear partial differential equations of the second order.

Between three such functions there always exists an algebraic relation, — a property corresponding to Weierstrass's first theorem concerning periodic functions (§ 4, end), and these functions serve to uniformize such an algebraic configuration.

Double integrals on the corresponding algebraic configuration§ are studied, being uniformized as functions of u, v, and in this investigation we have a forerunner of Picard's researches on algebraic functions of two variables, to which we shall presently turn.

Functions of the classes hitherto treated, namely, those which admit a group of transformations of type (1), are called *hyperfuchsian functions.*‖ The definition is not restricted to functions

* *C. R.*, 90 (1880), pp. 296, 731; *Journ. de Math.* (3), 8 (1882), p. 173.

† *C. R.*, 90 (1880), pp. 1118, 1267; *Ann. Ec. Norm.* (2), 10 (1881), p. 305.

‡ *C. R.*, 96 (1883), p. 320; *C. R.*, 99 (1884), p. 852. We note here a paper by Poincaré, *C. R.*, 94 (1882), p. 840, in which automorphic functions of two variables are obtained from the theory of numbers. Cf. also papers by Picard, *Acta*, 1 (1883), p. 297; ibid., 5 (1884), p. 121; *Ann. Ec. Norm.* (3), 2 (1885), p. 357.

§ Cf. III; § 1.

‖ Picard, *Acta*, 5 (1884), p. 121.

ential equations of the second order, the coefficients being polynomials in x and y, at most of the third degree, with integral coefficients.

These equations admit three linearly independent solutions, ω_1, ω_2, ω_3. If the latter be suitably chosen and their ratios set equal to two new variables,

$$\frac{\omega_2}{\omega_1} = u, \qquad \frac{\omega_3}{\omega_1} = v,$$

then these equations define x and y as single-valued functions of u, v. The domain of definition, D, is that part of the four-dimensional space of the variables $u = u' + iu''$, $v = v' + iv''$, in which

$$2v' + u'^2 + u''^2 < 0.$$

The proof is given by means of the solution of Jacobi's problem of inversion for $p = 3$; cf. § 2.

Picard shows that the functions thus obtained admit a properly discontinuous group of linear transformations of the type (1) which carry D over into itself, the coefficients being of the form $k + l\lambda$, where k and l are integers, and λ is a complex cube root of unity. These transformations are closely related to those of a ternary group:

$$X = M_1 x + P_1 y + R_1 z,$$

$$Y = M_2 x + P_2 y + R_2 z,$$

$$Z = M_3 x + P_3 y + R_3 z,$$

—the coefficients here being also rational functions of λ,—which leave the Hermiteian form

$$x\bar{x} + y\bar{y} + z\bar{z}$$

unchanged, where $\bar{x}$ denotes, as usual, the conjugate of x.

Generalizations of Riemann's P-Function. The investigations on which we have just reported suggest, through the hypergeometric integral and the hypergeometric differential equation

Blumenthal.* The group is that in which

$$x'_l = \frac{\alpha_l x_l + \beta_l}{\gamma_l x_l + \delta_l}, \qquad l = 1, \cdots, n,$$

the coefficients being taken as follows. An algebraic domain of rationality is assumed as given, $R = k$, where k denotes a root of an irreducible algebraic equation in the natural domain, $R = 1$. Furthermore, all the roots $k, k', \cdots, k^{(n-1)}$, shall be real. The coefficients $\alpha_1, \cdots, \delta_1$ are taken in $R = k$, and the coefficients $\alpha_l, \cdots, \delta_l$ are the corresponding numbers of the domain $R = k^{(l-1)}$. Finally, $\alpha_1\delta_1 - \beta_1\gamma_1$ is a totally positive unit of the domain $R = k$.

The subject of automorphic groups in one and more variables has been treated systematically by Fubini.†

§ 7. Algebraic Functions of Two Variables

The impetus given to the study of the algebraic plane curves and the geometry on them, through the researches of Plücker, Cayley, and Clebsch, in connection with the theory of the algebraic functions and the Abelian integrals as developed by Riemann, early made itself felt in the study of algebraic surfaces and algebraic functions of two variables. Thus we find a paper by Clebsch‡ of the year 1868, in which he discovers an invariant of an algebraic surface analogous to the deficiency p of an algebraic curve. The latter invariant may be defined as the number of essential constants in the general integral of the first kind, i. e., in the everywhere finite integral, and this integral can be written in the form

$$\int \frac{Q(x, y)}{f_y} dx, \qquad f_y = \frac{\partial f}{\partial y},$$

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* Cf. preceding reference. Furthermore Hecke, Göttinger Dissertation, 1910.

† Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe, 1908.

‡ *C. R.*, 67 (1868), p. 1238. Clebsch had only the adjoint Q's of degree $m - 4$. The everywhere finite double integral is due to Noether, *Math. Ann.*, 2 (1870), p. 293.

for which D is the hypersphere, but includes at least all functions admitting a properly discontinuous group of type (1) and meromorphic in a domain D defined by a relation

$$g(u', u'', v', v'') < 0,$$

where g is a quadratic polynomial. Moreover, the functions cannot be continued analytically beyond D.

In this same year Picard* began the investigation of functions which admit a group of transformations of type (2). These functions he denoted as *hyperabelian functions,* since the first problem which he was led to study concerning them was one related to the Abelian thetas and the Abelian modular functions, $p = 2$. The classes discussed yielded functions with properties analogous to those of the hyperfuchsian functions.

Generalizations. In a systematic development of the theory of the automorphic functions of several complex variables a question of first importance is that of the existence of a fundamental domain belonging to a properly discontinuous group. A solution of this problem for such groups of projective transformations in n variables, — groups of type (1), — has been given by Hurwitz.†

The extension of the two theorems of Weierstrass, § 4, for the case of automorphic functions in n variables has been treated by Wirtinger‡ by the aid of methods of the general theory of functions.

A systematic generalization of the theory of a class of hyperabelian functions was outlined by Hilbert and elaborated by

* Notes in the *Comptes Rendus* for 1884, followed by a systematic presentation in *Journ. de Math.* (4), 1 (1885), p. 87. Cf. further Bourget, *Toulouse Ann.*, 12 (1898), p. D 1; Humbert, *Journ. de Math.* (5), 5, 6, 7, 9, 10 (1899–1904), and (6), 2 (1906).

† *Math. Ann.*, 61 (1905), p. 325.

‡ *Sitzungsber. der Wiener Akad.*, 108 (1899), p. 1239. For the special case of hyperabelian functions of n variables cf. Blumenthal, *Math. Ann.*, 56 (1903), p. 510; ibid., 58 (1904), p. 497. Picard had long since used the second theorem, stated for automorphic functions of two variables; cf. *Journ. de Math.* (4), 1 (1885), p. 313.

The Second Deficiency. There is a second numerical invariant which can be defined as follows. Consider the linear family of adjoint surfaces of degree $m - 4$:

$$Q(x, y, z) = \alpha_1 Q_1 + \alpha_2 Q_2 + \cdots + \alpha_{p_g} Q_{p_g}.$$

These surfaces cut the ground surface $f = 0$ in certain fixed curves,—including always the multiple curves of f,—and a variable curve, l. This latter curve will, in general, be irreducible, and we assume the non-specialized case. It is a twisted space curve, and it has, as such, a definite deficiency, which can be defined, for example, as the deficiency of the Riemann's surface corresponding to the curve. This deficiency is the same in general for the different curves of the family, and it is this number, $p^{(1)}$, which is called the *second* or *numerical deficiency*,—Kurvengeschlecht,* le second genre. It is an invariant under the group of birational transformations, (A).†

The Line Integral. There is another generalization of the Abelian integrals possible for the algebraic functions of two variables, namely,‡

$$\int P\,dx + Q\,dy,$$

where P and Q are rational functions of x, y, z, the third variable being a root of the irreducible algebraic equation $f(x, y, z) = 0$, and where, moreover, the condition of integrability is satisfied:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

* This invariant is due to Noether, *Math. Ann.*, 8 (1875), p. 520.

† Cf. Picard et Simart, l. c., p. 206. Noether introduced a further invariant, $p^{(2)}$, namely, the number of variable points of intersection of two curves l. In general, $p^{(2)} = p^{(1)} - 1$, but for special surfaces $p^{(2)} < p^{(1)} - 1$. Cf. Picard et Simart, ibid., p. 209.

‡ Picard, *Journ. de Math.* (4), 1 (1885), p. 281; ibid. (4), 5 (1899), p. 135. The latter paper is the memoir to which the prize of the Paris Academy of Sciences was awarded. It forms the foundation of the later presentation of the theory of Picard and Simart, Fonctions algébriques de deux variables, Paris, 1897–1900.

where $f(x, y) = 0$ is the equation of the ground curve C_m, assumed irreducible, and $Q(x, y)$ is an *adjoint* polynomial of degree $m - 3$. If, in particular, C_m has only ordinary double points, $Q = 0$ is any C_{m-3} that passes through these points.

Consider now an irreducible algebraic surface $f(x, y, z) = 0$ of degree m with only ordinary multiple lines and isolated multiple points. Then the double integral (II, § 2)

$$\int\int \frac{Q(x, y, z)}{f_z} dxdy$$

taken over an arbitrary regular surface, open or closed, lying in the four-dimensional Riemann manifold corresponding to the function z of x, y defined by the equation $f = 0$, will remain finite provided $Q(x, y, z) = 0$ is an *adjoint* surface of degree $m - 4$, i. e., a surface which passes through the multiple lines and has a multiple line of order $k - 1$ at least in every multiple line of f of order k; and which moreover has a multiple point of order $q - 2$ at least in every isolated multiple point of f of order q.* Such an integral is called a *double integral of the first kind.* The number of linearly independent integrals of this class, i. e., the number of essential constants in the adjoint polynomial $Q(x, y, z)$ is called the *deficiency*, or more precisely, the *geometrical deficiency*,—Flächengeschlecht,† genre géometrique,—in distinction from the numerical deficiency presently to be considered, and is denoted by p_g. It is an invariant under the group of birational transformations of the surface:

$$(A) \qquad \left.\begin{aligned} X &= r_1(x, y, z), \\ Y &= r_2(x, y, z), \\ Z &= r_3(x, y, z), \end{aligned}\right\} \qquad \left.\begin{aligned} x &= R_1(X, Y, Z), \\ y &= R_2(X, Y, Z), \\ z &= R_3(X, Y, Z). \end{aligned}\right\}$$

In case the surface f has no multiple lines or points,

$$p_g = \frac{(m - 1)(m - 2)(m - 3)}{6}.$$

* Cf. Picard et Simart, Fonctions algébriques de deux variables, vol. 1, 1897, ch. 7; in particular, p. 189.

† The invariant is due to Clebsch; the name to Noether.

admit two-dimensional cycles, and it is these that form the analogue of the linear cycles in the case of the algebraic functions of a single variable. With these are connected the double integrals of Noether.

The methods employed in these early geometric investigations are largely those of intuition and analogy. Picard recognizes this fact, but points out that his chief object was to throw light on a theory at that time wholly new.

Geometry on Algebraic Curves and Surfaces. The purely algebraic theory of the geometry of systems of points on algebraic curves has been extended to algebraic surfaces and systems of curves lying on them.*

The Point of View of the Theory of Numbers. The methods of the theory of algebraic numbers, first extended to the algebraic functions of a single variable, have been used by Hensel† for the study of algebraic functions of two variables. In his treatment of the theory of the algebraic functions of a single variable Weierstrass had used purely algebraic methods. Hensel describes his own methods for algebraic functions of two variables as the direct generalization of Weierstrass's methods.

In a preliminary study of these functions Hensel deduces series developments which apply to the neighborhood of a branch-line or of a multiple-line of the surface. The form of the development in the neighborhood of a finite point, which we will take as the origin (0, 0, 0), is the following:

$$z = e_1(x)(y - y_0)^{1/b} + e_2(x)(y - y_0)^{2/b} + \cdots,$$

where b is a positive integer. The coefficients $e_k(x)$ and the variable y_0 are analytic functions of ξ, where

$$\xi = x^{1/a}$$

and α is a positive integer. In fact, the equation of the branch

* Noether, *Math. Ann.*, 2 (1870), p. 293; ibid., 3 (1871), pp. 161, 547; ibid., 8 (1875), p. 495. Picard et Simart, Fonctions algébriques de deux variables, vol. 2.

† *Acta*, 23 (1900), p. 339; *Jahresber. D. M.-V.*, 8 (1899), p. 221.

Such an integral is a function of two independent variables, and these may be taken as x, y or y, z or z, x.

A division of such integrals into three classes, corresponding to the three classes of Abelian integrals, at once suggests itself. In the first paper above referred to Picard studies the integrals of the first class, namely, the everywhere finite integrals.* He finds here a situation diametrically opposite to that in the case of the Abelian integrals. If $f(x, y) = 0$ is an irreducible algebraic equation of degree greater than 2, there will in general exist integrals of the first class corresponding to it; it is only when the curve is highly specialized (unicursal) that this is not the case.

To the non-specialized algebraic surface of arbitrary degree, however, there correspond no integrals of the first kind with the trivial exception of a constant. A special class of surfaces and integrals is treated, the former being those which can be uniformized by means of quadruply periodic functions of two independent variables.

It was in these papers that Picard began the study of questions relating to the connectivity of the surfaces which present themselves. The points of an algebraic surface fill a four-dimensional region,—be that region assumed as a four-dimensional manifold in space of six or more dimensions, or as a multiple-sheeted Riemann manifold, or as a fundamental domain, for which the parallelogram of periods is the prototype. In this four-dimensional manifold the linear cycles (closed curves) and the two-dimensional cycles (closed surfaces) are of especial importance. Picard finds the striking result that, in the case of a non-specialized algebraic surface, any linear cycle can be drawn together continuously to a point. This fact explains,—or is explained by,—the non-existence of integrals of the first class on such a surface.

On the other hand, a non-specialized algebraic surface does

* Picard's first publication relating to the integrals of the second class appeared in the *Comptes Rendus*, 100 (1885), p. 843.

LECTURE II

SOME GENERAL THEOREMS

§ 1. Definitions and Elementary Theorems

Let

$$F(z_1, \cdots, z_n)$$

be a complex function of the n complex variables

$$z_k = x_k + i\,y_k, \qquad k = 1, 2, \cdots, n,$$

which is defined uniquely at each point of a $2n$-dimensional continuum T. Of the two current definitions mentioned in I, § 1, we will choose the second and say: F is *analytic in* T if, at every point of T, it admits a derivative with respect to each of the complex arguments $z_1, \cdots, z_n$ and if, furthermore, it is continuous in T. The latter condition turns out to be a consequence of the former, cf. § 5, and may, therefore, be stricken from the definition. But it is better to retain it for a time, since it suffices for a simple proof of the integral theorems, and with the aid of these all the principal theorems are readily established.

The function $F(z_1, \cdots, z_n)$ is said to be *analytic in a point* $(a_1, \cdots, a_n)$ if it is analytic throughout some region T containing the point in its interior. Similarly, F is said to be *analytic in a manifold* M of one or more dimensions if it is analytic throughout a region T containing M in its interior. If M is closed, i. e., if M contains its boundary points, then, for F to be analytic in M, it is clearly sufficient that F be analytic in every point of M.

The Cauchy-Riemann Differential Equations. The differential equations which the real part u (or the coefficient v of the pure imaginary part) of an analytic function satisfies are the following:

$$\frac{\partial^2 u}{\partial x_k \partial x_l} + \frac{\partial^2 u}{\partial y_k \partial y_l} = 0, \qquad \frac{\partial^2 u}{\partial x_k \partial y_l} - \frac{\partial^2 u}{\partial y_k \partial x_l} = 0.$$

of the discriminant under consideration is

$$y = y_0(x) = \beta_1 x^{1/a} + \beta_2 x^{2/a} + \cdots.$$

§ 8. Analysis Situs

In closing we refer briefly to the subject of analysis situs in the geometry of n dimensions. Riemann was the first to recognize the importance of this subject for the surfaces which bear his name. He had also thought about the problem for higher manifolds.* Betti† considered the simple closed cycles of one dimension (curves), of two dimensions (surfaces), and, generally, of m-dimensions, $m = 1, 2, \cdots, n - 1$, which can be described in the n-dimensional region under consideration, and he introduced the numbers called after him, which indicate how many cycles of a given class are needed as a basis to represent a general cycle of that class.

Attention has already been called to Picard's work on questions in this field relating to algebraic surfaces, § 7.

Poincaré perceived the value of this branch of geometry for analysis and published a series of papers on the subject.‡ Following Betti, he considered integrals extended over closed m-dimensional manifolds (cycles) in the n-dimensional region, and he found the conditions that the value of the integrals be invariant of a restricted deformation of the manifold; II, § 2. Such integrals may form the basis for determining the Betti numbers.§

* Cf. the fragment in his collected works, Werke, 1 ed., p. 448; 2d ed., p. 479.

† *Annali di mat.* (2), 4 (1870–71), p. 140.

‡ Cf., in particular, *Journ. Éc. Polytech.* (2), Cah. 1 (1895), p. 1; also the account given in Picard et Simart, Fonctions algébriques de deux variables, vol. 1, ch. 2.

§ An elementary geometric treatment of the analysis situs of hypermanifolds has recently been given by Veblen and Alexander, *Annals of Math.* (2), 14 (1913), p. 163.

It is sometimes useful to think of a B-region as a rectangle, when $n = 2$, or as a parallelepiped, when $n = 3$, or as a prismatoid in higher space, just as we picture curves and surfaces to ourselves in the plane or in space, even when the coordinates are complex; the point being that the variation to which each coordinate is subject is independent of that to which any other coordinate is subject.

Cauchy's Integral Formula. A first form for this formula is that suggested by a B-region. Let $F(z_1, \cdots, z_n)$ be analytic in a region T, and let B be a regular region lying in T. For convenience, let $n = 2$. It is at once seen that F is given for any point interior to B by the formula:

$$(1) \qquad F(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{C_1} \frac{dt_1}{t_1 - z_1} \int_{C_2} \frac{F(t_1, t_2)}{t_2 - z_2} dt_2,$$

where C_k denotes the boundary of B_k and the integral is extended in the positive sense.

This iterated integral suggests readily a double integral, extended over a surface, i. e., a two-dimensional manifold, which lies in the boundary of B. But the complete boundary of B is a $2n - 1 = 3$-dimensional manifold, and so the analogy with Cauchy's Integral Formula for $n = 1$:

$$(2) \qquad F(z) = \frac{1}{2\pi i} \int_C \frac{F(t)}{t - z} dt,$$

is in so far only partial, that the earlier integral (2) is extended over the *complete* boundary of the region for z, whereas the present integral (1) is extended over a manifold of lower order which lies in the boundary.

§ 2. Line and Surface Integrals, Residues, and their Generalizations

In space of three dimensions the surface integral of a continuous real function, $f(x, y, z)$, extended over a curved surface Σ, is defined in the familiar manner:

Thus, when $n = 2$, there are four equations:*

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} = 0, \qquad \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} = 0,$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0, \qquad \frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial y_1 \partial x_2} = 0.$$

Cylindrical and B-Regions. A general region of hyperspace can be described analytically by one or more inequalities. Thus the interior of the hypersphere of radius r, with its centre at the origin, is given by the inequality

$$x_1^2 + x_2^2 + \cdots + x_m^2 < r^2,$$

or by the pair of inequalities:

$$-\sqrt{r^2 - x_1^2 - \cdots - x_{m-1}^2} < x_m < \sqrt{r^2 - x_1^2 - \cdots - x_{m-1}^2}.$$

A particularly simple and important class of regions in the space of the n complex variables $z_1, \cdots, z_n$ is the following. Let T_k, $k = 1, \cdots, n$, be an arbitrary two-dimensional continuum in the complex z_k-plane, and let z_k be any one of its points. Then the region of $2n$-dimensional space whose points $(x_1, y_1, x_2, \cdots, y_n)$ correspond to $z_1, \cdots, z_n$ is called a *cylindrical region*, and may be denoted by $(T) = (T_1, \cdots, T_n)$, or, more simply, by T. It is a continuum, and so consists only of interior points.

Let B_k, $k = 1, \cdots, n$, be a regular region of the z_k-plane, i. e., a finite continuum plus its boundary, the latter consisting of a finite number of regular curves having a finite number of multiple points and points of intersection; and let z_k be any point of B_k. Then the corresponding region of the $2n$-dimensional space shall be called a *regular cylindrical region* or a *B-region*; and it shall be represented as $(B) = (B_1, \cdots, B_n)$, or, more simply, as B.

The boundary of a cylindrical region is composed of those points $(z_1, \cdots, z_n)$ for which at least one z_k lies on the boundary of its T_k or B_k. It consists of a single piece, and is a manifold of $2n - 1$ dimensions.

* Poincaré, *C. R.*, 96 (1883), p. 238; *Acta*, 2 (1883), p. 99; ibid., 22 (1898), p. 112.

integral may be interpreted as a residue, the particular surface over which the integration is extended lying in the boundary of the cylindrical region B. It appears, however, on comparison with Poincaré's criteria that the surface may be deformed continuously without altering the value of the integral, and hence we are led to a generalization of the integral formula. The integral appears as a residue, the surface of integration being thought of as a closed surface which is linked in a certain way with the two singular surfaces

$$t_1 - z_1 = 0 \quad \text{and} \quad t_2 - z_2 = 0 \tag{3}$$

after the fashion of a closed curve in space of three dimensions which is linked with certain right lines of that space. For, the singular surfaces (3) are manifolds of order 2, not 3, in space of 4 dimensions, and so they do not cut that space in two.

Poincaré establishes the following theorem. If

$$R(w, z) = \frac{H(w, z)}{G(w, z)}$$

is a rational function of w, z, and if

$$\int\int R(w, z)\, dw\, dz$$

is extended over any regular closed surface which has no point in common with the singular manifold $G(w, z) = 0$, then the above integral can be evaluated in terms of the moduli of periodicity of Abelian integrals belonging to the algebraic configuration or configurations

$$G(w, z) = 0.$$

He also considers the case that the surface meets the singular manifold, and obtains here an evaluation in terms of Abelian integrals with variable limits of integration.

§ 3. The Space of Analysis, and Other Spaces

The *space of analysis* can be defined as coextensive with the point set $\{(z_1, \cdots, z_n)\}$, where each one of the complex variables

$$\int\!\!\int_{\Sigma} f(x, y, z)\, d\Sigma = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta\Sigma_k,$$

the element of surface, $\Delta\Sigma_k$, being taken as an essentially positive quantity.

Allied with this integral is the other surface integral:

$$\int\!\!\int_{\Sigma} A\, dy\, dz + B\, dz\, dx + C\, dx\, dy,$$

where A, B, C are continuous functions of x, y, z. Here, the sign to be attached to the differential factor, $dy\, dz$, etc., requires special definition, and can be assigned in terms of the sense of an indicatrix which moves continuously over the surface, or in terms of the signs of the Jacobians

$$\frac{\partial(y, z)}{\partial(s, t)}, \qquad \frac{\partial(z, x)}{\partial(s, t)}, \qquad \frac{\partial(x, y)}{\partial(s, t)},$$

where s, t are parameters by means of which x, y, z are expressed.

Such integrals, suitably generalized for manifolds of order k in space of m dimensions, have been applied by Picard and Poincaré* to analytic functions of n complex variables. Poincaré develops conditions that the value of such an integral, extended over a closed manifold, be invariant of slight deformations of the manifold, and hence also of large variations, provided the manifold retains its character and does not sweep over a point in which an integrand is discontinuous or the conditions in question cease to hold. Cf. also I, § 8.

Residues. The value of any such integral Poincaré calls a *residue*. Only bilateral manifolds come into consideration, since an integral extended over a unilateral manifold evidently vanishes.

As a first application of the foregoing, consider Cauchy's integral formula. In the form in which it stands above, the

* Picard, *C. R.*, 96 (1883), p. 320; ibid., a series of papers in vols. 102–3 (1886). Poincaré, *C. R.*, 102 (1886), p. 202; *Acta*, 9 (1887), p. 321, where reference to Jacobi and Marie in connection with these integrals is made.

$$z = \frac{1}{z'}, \qquad f(z) = \varphi(z'),$$

the function $\varphi(z')$, which is not defined in the point $z' = 0$, but is analytic in the rest of the neighborhood of this point, shall have a removable singularity in the point $z' = 0$.

Returning to functions of several variables, let us raise again the question, why introduce the space of analysis? A contribution toward an answer to this question is to be found in the two theorems of § 4, below. For simplicity, let us restrict ourselves to the first one. This theorem is not true if our hypothesis be merely that the function shall be meromorphic in every point of finite space. Some further hypothesis relating to its behavior at infinity, or to the behavior of the function when subjected to certain transformations, is essential. And now Weierstrass supplied this condition,—or appears to have done so,—in the way indicated above.

But is this the only way in which this end can be attained without doing violence to simplicity or custom? By no means, as we shall presently see.

Projective Space and the Space of the Homogeneous Variables.—The space most familiar to the geometers is projective space, and this space is mapped in a $(1, \infty)$-fold manner on the space of $n + 1$ homogeneous variables $x_0, x_1, \cdots, x_n$. This latter space is the whole finite space whose points are $(x_0, x_1, \cdots, x_n)$, where each coordinate ranges over its whole finite Gauss plane, the one point $(0, 0, \cdots, 0)$ being excluded. We will speak of it as *the space of the homogeneous coordinates.*

The functions considered in this space had their origin in projective space (itself but an amplification of an ordinary finite space), and are homogeneous in the $n + 1$ variables,—polynomials, algebraic functions, and such transcendental functions as are suggested by the names of Aronhold, Clebsch and Gordan, Klein, and their school.

Might it not have been possible to choose the complementary hypothesis in Weierstrass's theorem is § 4, not with reference

z_k ranges over its extended plane,—the Neumann sphere. A point of that space lies *at infinity* if at least one of its coordinates is at the north pole of its sphere. A function $f(z_1, \cdots, z_n)$ is said to be *continuous, analytic,* or *meromorphic* at a point of the infinite region if, when each coordinate z_k which becomes infinite is replaced by a new point by means of the transformation

$$z_k' = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k}, \qquad \begin{vmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{vmatrix} \neq 0, \qquad \gamma_k \neq 0,$$

the transformed function is continuous, analytic, or meromorphic, —as the case may be,—at the transformed point.

What are some of the reasons for this extension of proper (finite) space? First, it is natural. In the case of analytic functions of a single complex variable the reasons are well known why it is desirable to extend the proper Gauss plane by a single point,—the point ∞. What more natural, then, than to take as the space of analytic functions of n complex variables the space defined by the n spheres of the individual variables?

But this reason is superficial. It is formal. The real object of extending proper (finite) space at all is to secure theorems which include among their hypotheses some requirement relating to the behavior of the function when one or more of its arguments become infinite. It is not essential that ideal elements,—*points at infinity,*—be introduced. The requirements can be stated in terms of a transformation, usually linear, though not necessarily projective, applied to the points of space proper, and the behavior of the transformed function in the neighborhood of a point or points for which the latter function is not defined.

Thus a function of a single complex variable, $f(z)$, can be defined as analytic at infinity without introducing any ideal element whatever if we proceed in either one of the following ways. In both cases we shall demand that $f(z)$ be analytic outside of a certain circle in the z-plane, and finite along this circle. And now we require further either (a) that $f(z)$ remain finite in the above region; or (b) that, if we set

the irreducible equation $f(x, y, z) = 0$, it is possible to associate with this function the surface in projective space given by setting

$$x = \frac{x_1}{x_0}, \qquad y = \frac{x_2}{x_0}, \qquad z = \frac{x_3}{x_0}.$$

It is, however, also possible to put

$$x = \frac{x_1}{x_0}, \qquad y = \frac{x_2}{x_0}, \qquad z = \frac{z_1}{z_0};$$

and still again to set

$$x = \frac{x_1}{x_0}, \qquad y = \frac{y_1}{y_0}, \qquad z = \frac{z_1}{z_0}.$$

There is another geometry that is well known, — the geometry of reciprocal radii, or the geometry of inversion. It would, of course, be a proceeding entirely coordinate with that which has been set forth above to extend the finite space of n complex variables to the space of that geometry.

These questions could not arise in the case of analytic functions of a single complex variable, for there the infinite region of projective geometry, the geometry of inversion, and the space of analysis are the same, namely, one point.

For the case of two complex variables, the infinite region of the space of analysis and the infinite region of projective geometry are different, and moreover the space of analysis and projective space can no longer be transformed on each other in a one-to-one manner and continuously. But the space of analysis is transformable in a one-to-one (but non-real) manner, and continuously, on the space of the geometry of inversion. When the number of complex variables exceeds two, all three spaces are distinct.*

* For a detailed treatment of these questions cf. a paper by Bôcher, *Bulletin Amer. Math. Soc.* (2), 20 (1914), p. 185. We note that the infinite region of the space of analysis consists of n complex $(n-1)$-dimensional manifolds (hyperplanes) which have as their sole common point the point $(\infty, \infty, \cdots, \infty)$.

to the space of analysis, but in terms of projective space? More precisely, we should demand, as before, that the function be meromorphic (III, § 2) in all points of the proper space of the variables $(z_1, \cdots, z_n)$ and we should then add the following hypothesis: Let a transformation of the type

$$z_k' = \frac{\alpha_0^{(k)} + \alpha_1^{(k)} z_1 + \alpha_2^{(k)} z_2 + \cdots + \alpha_n^{(k)} z_n}{\alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_n z_n},$$

$$\Sigma \pm \alpha_0 \alpha_1^{(1)} \alpha_2^{(2)} \cdots \alpha_n^{(n)} \neq 0,$$

$$0 < |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|,$$

be performed; and let $(a_1', \cdots, a_n')$ be a finite point not corresponding to any finite point of the $(z_1, \cdots, z_n)$ space. Then, if the transformed function be defined in each removable singularity of the neighborhood of $(a_1', \cdots, a_n')$ as equal to the limiting value which it approaches in that point, the thus extended function shall be analytic or meromorphic in $(a_1', \cdots, a_n')$.

Will the function $f(z_1, \cdots, z_n)$ under these conditions be rational? The answer is affirmative.* And the corresponding theorem holds for the algebraic case of § 4 also. What reason is there, then, for preferring the space of analysis to projective space as the space in which monogenic analytic configurations are to be studied?

The answer is that, so far as these two fundamental theorems are concerned, there is none. We shall have two theories of algebraic functions of several variables, and of the related transcendental functions, — and, more generally, of monogenic analytic functions, — according as we extend finite space in the one way or the other. Moreover, when $n > 2$, the choice is still larger, for the variables may then be divided into two classes, those of one class being transformed projectively, and those of the other class spherically, i. e., so that the Neumann sphere of each variable goes over into a Neumann sphere of a new variable. Thus, if z is defined as an algebraic function of x, y by

* Osgood, *Transactions Amer. Math. Soc.*, 13 (1912), p. 159.

Theorem. Let $f(x, y)$ be defined throughout a cylindrical region (S, S'), § 1. Let $f(x, b)$ be analytic in S, for every choice of b in S'; b, when once chosen, to be held fast. Similarly, let $f(a, y)$ be analytic in S', a being any point of S. Then $f(x, y)$ is analytic in the two independent variables x, y throughout the region (S, S').

The theorem is readily proven if the further hypothesis be added that the function remain finite, and under this restriction in sufficiently general for many of the cases which arise in practice.* It is, however, of distinct interest to know that the more general theorem is true. This latter result has been established by Hartogs.†

The theorem holds for functions of any number of variables.

Further theorems of the character of those here considered are given in the next paragraph, Theorems A, B.

§ 6. Sufficient Conditions that a Function be Rational or Algebraic

Hurwitz's proof of Weierstrass's theorem, § 4, yields more than is contained in the statement of that theorem. By means of it the following theorems can be established.

Theorem 1. If $f(z_1, \cdots, z_n)$ is meromorphic at every point of the coordinate axes; i. e., in each of the points

$$(0, \cdots, 0, z_k, 0, \cdots, 0), \qquad k = 1, \cdots, n,$$

where the variable z_k ranges over the whole extended z_k-plane, then $f(z_1, \cdots, z_n)$ is a rational function of its arguments.

This theorem can be stated in the following form.

Theorem 1′. If $f(z_1, \cdots, z_n)$ is meromorphic in each of those points of the infinite region which corresponds to any $n - 1$ north poles combined with any point whatever of the nth sphere, then the function is rational.

A special case of this theorem is the following.

* This theorem was proven by the author, *Math. Ann.*, 52 (1899), p. 462

† *Math. Ann.*, 62 (1905), p. 1. Cf. also Osgood, ibid., 53 (1900), p. 461.

§ 4. Rational and Algebraic Functions

To Weierstrass is due the theorem that a function of n complex variables which is meromorphic at every point of the space of analysis is a rational function.*

Weierstrass did not define the space in which the function is considered. He said "im ganzen Gebiete seiner Veränderlichen." It appears, however, from more explicit statements in similar cases† that he thought of each variable as an arbitrary point of its extended plane.

A similar theorem holds for algebraic functions. If a function of n complex variables is finitely multiple-valued and if, in the neighborhood of every point of the space of analysis, the values of the function can be so grouped as to satisfy one or more algebroid relations,

$$A_0 w^m + A_1 w^{m-1} + \cdots + A_m = 0,$$

where the A's are analytic in the point in question,—and to be exhausted in said neighborhood by these systems,—then the function is algebraic.

§ 5. Sufficient Conditions that a Function of Several Complex Variables be Analytic

In order that a function of two real variables be analytic it is not enough that the function be analytic in each variable separately when the other is held fast, as is shown by the example:

$$f(x, y) = \frac{xy}{x^2 + y^2}, \qquad 0 < |x| + |y|;$$

$$f(0, 0) = 0,$$

the function being considered in the neighborhood of the origin. When, however, we allow the variables to take on complex values, the case stands otherwise.

* *Journ. für Math.*, 86 (1880), p. 5 = Werke 2, p. 129. The theorem was proven by Hurwitz, *Journ. für Math.*, 95 (1883), p. 201.

† Cf. for example Werke, 3, p. 100, 7th line from end.

$n-1$ north poles combined with any point whatever of the nth sphere, then f is a constant.

As a special case of the theorem we have the

Corollary. If $f(z_1, \cdots, z_n)$ is analytic in every point of the infinite region of the space of analysis, then f is a constant.

This last result can be stated in a form wholly independent of any assumption regarding the infinite region.

Theorem B. If $f(z_1, \cdots, z_n)$ is analytic at all finite points outside a fixed hypersphere:*

$$G < x_1^2 + y_1^2 + x_2^2 + \cdots + y_n^2,$$

and if f is finite in this region, then f is a constant.

Returning now to theorems, relating to rational functions, we have the following.

Theorem 2. If $f(z_1, \cdots, z_n)$ is a rational function of each individual variable, when all the others are assigned arbitrary values in the neighborhood of a certain fixed point and then held fast, then f is rational in all its arguments.

The proof of Theorem I is covered by Hurwitz's reasoning, and the same is true of Theorem II, provided the additional hypothesis is made that the function be analytic in all its arguments in the neighborhood of the fixed point in question. In practice, this further condition appears usually to be fulfilled. For a proof that this condition is a consequence of the others I am indebted to Professor E. E. Levi.

Both theorems can be extended to algebraic functions, the hypothesis then being that the function is N-valued, and that, moreover, it is algebroid, where before it was meromorphic.

§ 7. On the Associated Radii of Convergence of a Power Series

Let

$$\sum C_{\nu_1 \cdots \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n}$$

be a power series convergent for a set of values of the arguments,

* This hypothesis may equally well be written in the form

$$G < |z_1| + \cdots + |z_n|.$$

Corollary. If $f(z_1, \cdots, z_n)$ is meromorphic in every point of the infinite region of the space of analysis, then f is a rational function of all its arguments.

These theorems readily suggest others, in which the word *meromorphic* is replaced in the hypothesis by *analytic;* the conclusion then being that the function is a constant.

Theorem A. If $f(z_1, \cdots, z_n)$ is analytic in every point of the coordinate axes, then f is a constant.

The manifold M consisting of the coordinate axes is perfect, and hence f is analytic in a $2n$-dimensional region T enclosing the axes. It is possible, in particular, to choose a positive number h so that f is analytic in the region

$$|z_1| < h, \quad \cdots, \quad |z_{k-1}| < h, \quad |z_{k+1}| < h, \quad \cdots, \quad |z_n| < h,$$

z_k ranging over the whole extended z_k-plane; $k = 1, \cdots, n$.

Consider f in the region

$$\Sigma: \qquad |z_k| < h, \qquad\qquad k = 1, \cdots, n.$$

Let $(a_1, \cdots, a_n)$ be a point of this region. The function

$$f(a_1, \cdots, a_{n-1}, z_n)$$

is analytic over the whole extended z_n-plane. Hence it is a constant. Hence

$$\frac{\partial f(z_1, \cdots, z_n)}{\partial z_n} = 0$$

in the point $(a_1, \cdots, a_n)$. But this was any point of Σ.

It appears, then, that

$$\frac{\partial f}{\partial z_k} \equiv 0, \qquad\qquad k = 1, \cdots, n,$$

and from this fact follows the truth of the theorem.

As in the case of Theorem 1, so here the theorem admits an alternative statement.

Theorem A′. If $f(z_1, \cdots, z_n)$ is analytic in those points of the infinite region of the space of analysis which correspond to any

sense, that they cannot be expanded for any one of the variables x_k without being contracted for others, yield associated radii of convergence.

Detailed Consideration of the Case $n=2$. The mutual relations between the r's have been studied extensively. Let the number of variables be two, and let the series be written:

(1) $$\Sigma C_{m,n}\, x^m y^n.$$

If, for a pair of values x_0, y_0, neither of which is zero, the terms of the series (1) remain finite, then it is well known that the series converges (and hence converges absolutely), when

$$|x| < r, \quad |y| < s,$$

where $r = |x_0|$, $s = |y_0|$.

Let r have an arbitrary value in the interval $0 < \xi < |x_0|$. To this value of r may correspond larger values of s,—in fact, there may be no limit to s. If, however, the latter is not the case, let $\varphi(r)$ denote the upper limit of s for the value of r in question. Then r and $\varphi(r)$ form a pair of associated radii of convergence. Also, $\varphi(r)$ is spoken of as *the associated radius of convergence* (i. e., associated with r as independent variable). If, for a given r, s has no upper limit, the associated radius of convergence is said to be *infinite*.*

A necessary and sufficient condition that r, s be associated radii of convergence has been obtained by Lemaire,† who generalized a familiar theorem of Cauchy's for power series in a single variable. It is as follows. Consider the points of condensation of the set of numbers

$$\sqrt[m+n]{|C_{m,n}|r^m s^n},$$

where m, n independently range over the positive integers and zero. Then the condition is that the points of the original set remain in the finite region, and that the point of the derived set most remote from the point 0 be situated at 1.

* This is not, of course, the same thing as saying that, for such a value of r, $\varphi(r)$, becomes infinite.

† *Bull. des Sci. Math.* (2), 20 (1896), p. 286.

no one of which is zero.* A set of positive numbers $r_1, \cdots, r_n$ such that the series converges when

$$|x_k| < r_k, \qquad k = 1, \cdots, n,$$

but diverges when

$$|x_k| > r_k, \qquad k = 1, \cdots, n,$$

is called a set of *associated radii of convergence.*

The numbers $r_1, \cdots, r_n$ are in general mutually dependent on each other. Thus in the case of the series

$$\Sigma x_1{}^\nu x_2{}^\nu = \frac{1}{1 - x_1 x_2}$$

it is clear that

$$r_1 r_2 = 1.$$

Geometric Interpretation. Geometrically the associated radii of convergence may be interpreted as follows. Denote by T_k the circle $|x_k| < \rho_k$ and by T the $2n$-dimensional cylindrical region $T = (T_1, \cdots, T_n)$.

Let $f(x_1, \cdots, x_n)$ be analytic at the origin. Then the ρ_k's can be so chosen that T lies in the region of definition of the element $f(x_1, \cdots, x_n)$ in question. And now let the ρ_k's increase. Any system of values

$$\rho_k = r_k, \qquad k = 1, \cdots, n,$$

such that the function is analytic in the corresponding region $T = (T_1, \cdots, T_n)$, but no one of the T_k's can be replaced by a larger circle without diminishing some other T_k and have this properly preserved, is a system of associated radii of convergence.

Thus we may picture to ourselves a variable cylindrical region T in the domain of definition of the monogenic analytic function $f(x_1, \cdots, x_n)$. Those regions T that reach out to singular points of the function and, moreover, are maximum regions in this

* By a convergent multiple series $\Sigma u_{\nu_1 \ldots \nu_n}$ we mean a series such that every simple series formed from its terms converges. If, then, a multiple series converges, it necessarily converges absolutely.

Other multiple series have been investigated in recent years by Pringsheim and Hartogs.

be the function which corresponds to r as associated radius. Let

$$r_1 < r_2 < r_3$$

be three points of the interval of definition of $\varphi(r)$. Then

$$(2)\qquad \begin{vmatrix} 1 & \log r_1 & \log \varphi(r_1) \\ 1 & \log r_2 & \log \varphi(r_2) \\ 1 & \log r_3 & \log \varphi(r_3) \end{vmatrix} \leqq 0.$$

This is the relation designated by Hartogs as the *Fundamental Property*. It was proven by Fabry by means of Lemaire's theorem cited above. Hartogs gave several proofs, one of which is based on his function R_x defined below. He has also thrown Fabry's proof into exceedingly simple form.*

The theorem admits the following interpretation. Let

$$(3)\qquad x = \log r, \quad y = \log s.$$

Then

$$(4)\qquad \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \leqq 0.$$

Hence the curve that represents y as a function of x, the above curve (3), or

$$(5)\qquad y = \omega(x),$$

is always continuous† and concave downward.

Let the function

$$f(x, y) = \sum C_{m,n} x^m y^n$$

be analytic in the points $(x, 0)$, where $|x| < R$. If the associated radius corresponding to one single value of r, $r = r_1$, in the interval $0 < r < R$ is infinite, then the associated radius is infinite for every value of r in this interval. In this case, then, the function $\varphi(r)$ does not exist.

* *Jahresber. D. M.-V.*, 16 (1907), p. 232.

† This property was established by A. Meyer, *Stockholm Ved.-Ak. Förh. Öfv.*, 40 (1883), No. 9, p. 15, and Phragmén, ibid., No. 10, p. 17. Cf. the reference to Weierstrass, p. 148.

The corresponding condition holds for a power series in any number of variables.

The properties of the function

$$s = \varphi(r)$$

have been investigated, the most important of the results being the following.* First, some obvious properties.

If, for $r_0 > 0$, the associated radius of convergence is infinite, then it is infinite for every smaller value of r: $0 \leqq r \leqq r_0$.

As r increases, $\varphi(r)$ decreases or remains constant; i. e., $\varphi(r)$ is a decreasing monotonic function of r.†

The domain of definition of $\varphi(r)$ consists of an interval

$$0 \leqq R_0 < r < R_1 \quad \text{or} \quad 0 \leqq R_0 < r < \infty,$$

where, however, it is not obvious whether an extremity of the interval shall pertain to the interval or not.

The basal theorem relating to $\varphi(r)$ is the following.

Theorem.‡ Let

$$\sum C_{m,n} x^m y^n$$

be a double power series, and let

$$s = \varphi(r),$$

$$0 \leqq R_0 < r < R_1, \quad \text{resp.} \quad 0 \leqq R_0 < r < \infty,$$

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*The leading results here given were obtained by Phragmén as early as 1883 and published by him in a notable paper cited below. They were extended to n variables by A. Meyer, Thesis, Upsala, 1887.

† This property, together with the property that $\varphi(r)$ is continuous, was given by Weierstrass in his lectures; W. S., 1880/81. Cf. also the next reference.

‡ This theorem in its present form was first given by Fabry, *C. R.*, 134 (1902), pp. 1190, and rediscovered by Hartogs, Thesis, Munich, 1904 and Habilitationsschrift, 1904 = *Math. Ann.*, 62 (1905), p. 49 and p. 81. By the aid of it the theorems which follow in the text are easily proven; cf. Fabry and Hartogs, l. c.

A third paper closely related to the two just cited and containing a number of their results, obtained by a different method, was published by Faber, *Math. Ann.*, 61 (1905), p. 289. Faber's methods apply to power series with any number of variables, and his paper contains generalizations of theorems discussed in the text for the case of two variables.

$f(x, y)$ be analytic in the point $(x_0, 0)$; let

$$f(x, y) = \sum C_{m,n}(x - x_0)^m y^n;$$

and let r, $\varphi(r)$ be the associated radii of convergence corresponding to this series, if they exist. Then

$$R_{x_0} = \lim_{r=0} \varphi(r).$$

Thus R_{x_0} is defined for an arbitrary point $(x_0, 0)$; and if this point is designated merely as $(x, 0)$, we write, as a matter of notation, simply R_x.

The function R_x can also be obtained as follows. The function $f(x, y)$, analytic in $(x_0, 0)$, is analytic at all points (x_0, y) for which $|y|$ is duly restricted. The upper limit of the radius of the latter circle, if one exists, is the number R_{x_0}.

The geometric interpretation of a pair of complex numbers as a point of the plane of analytic geometry can here, too, be used with advantage. The series will then be thought of as converging throughout a certain rectangle with its centre at $(x_0, 0)$ and its sides parallel to the coordinate axes; such a rectangle to be a maximum rectangle in the sense that neither side may be increased without diminishing the other side. And now, as the base parallel to the x-axis approaches 0, the half-altitude, if it remains finite, approaches as a limit R_{x_0}.

Or, again, we may use the geometric interpretation in terms of the cylindrical regions of § 1.

It is obvious that, corresponding to any arbitrary point x_0 for which R_{x_0} exists, the function $f(x, y)$ has a singular point in some point (x_0, y_0) for which $|y_0| = R_{x_0}$.

The function R_x is real and positive, and it is semi-continuous in this sense. Let $x = x_0$ be any point in which it is defined, and let ϵ be an arbitrarily small positive number. Then there exists a positive δ such that

$$R_x \geqq R_{x_0} - \epsilon, \qquad |x - x_0| < \epsilon.$$

Let

$$\sum C_{m,n} x^m y^n$$

From this theorem it follows that, when the function $\varphi(r)$ exists at all, its interval of definition reaches back to the origin: $0 \leqq r < R$.

If the function $\varphi(r)$ exists and is constant in a portion of the interval of definition, then $\varphi(r)$ is constant clear back to the beginning of the interval.

The function $\omega(x)$ possesses a finite forward derivative and a finite backward derivative, neither of which is positive. The same is true of the function $s = \varphi(r)$.

If x_1 and x_2 are any two points of the interval of definition of $\omega(x)$:

$$-\infty < x_1 < x_2 < \log R,$$

neither of the above-named derivatives in the point x_2 exceeds either one of the derivatives in x_1.

From these results it is clear that, if P is any point of the curve (5), then a straight line whose slope is negative or nil can be drawn through P, such that the curve nowhere rises above the line.

By means of such lines,—"tangents," as Hartogs calls them,—Hartogs and Faber* show that the fundamental property (2) is the only condition which the function $\varphi(r)$ must fulfil. In other words: Let $\varphi(r)$ be any function of r which is defined throughout an arbitrary interval $0 \leqq r < R$, is positive there, and is subject to the condition (2). Then there exists a double power series

$$\sum C_{m,n} x^m y^n$$

to which the numbers r, s correspond as associated radii of convergence, where

$$s = \varphi(r), \qquad 0 \leqq r < R.$$

§ 8. Hartogs's Function R_x

In a number of his investigations Hartogs makes extended use of a function R_x which can be defined as follows.† Let

* Hartogs, *Math. Ann.*, 62 (1905), p. 84. Faber, l. c. Since Faber does not introduce the logarithm, the tangents appear as his W-curves.

† Hartogs, Dissertation, and *Math. Ann.*, 62 (1905), pp. 24, 25. The notation there used is R'_{x_0}.

§ 9. On the Analytic Continuation of a Logarithmic Potential

Let

$$w = f(z)$$

be an analytic function of the single complex variable z, and let

$$w = u + vi, \quad z = x + yi.$$

Then

$$u = \varphi(x, y)$$

is a logarithmic potential function of the real variables x, y. Moreover, as is well known, u is an analytic function of x, y. As such, it admits definition for complex values of the arguments, and thus gives rise to a monogenic analytic configuration in two independent variables.

For the real values of x and y for which the logarithmic potential was originally considered, u is real. It is, now, quite conceivable that we may be able to pass continuously, i. e., by analytic continuation through the complex domain, to another part of the analytic configuration in which (x, y), and also u, are real, and thus arrive at a new real solution u_1 of Laplace's equation, $\Delta u = 0$, not obtainable from the earlier one by analytic continuation along a real path. In particular, the function $f(z)$ may have a lacunary space, and it is conceivable that u_1 might be defined in that space.

Study* has considered this question, and has shown that the answer is negative. The only real solutions of Laplace's equation which can be obtained by analytic continuation are those which are obtainable by continuation along a path lying wholly in the real domain.

Study generalizes the question here considered and solves the corresponding problem, referring at the same time to a paper of Segre.†

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* *Math. Ann.*, 63 (1906), p. 240.

† Ibid., 40 (1892), p. 465, ll. 10–14.

be a double power series with the associated radii of convergence r, $\varphi(r)$. Consider the points x of the circle $|x| < r$, where r has an arbitrary value in the interval of definition of the function $\varphi(r)$. Then the lower limit of R_x for the points of this circle is equal precisely to $\varphi(r)$.

On the following theorem Hartogs bases his proof of a number of important theorems.*

Theorem. Let T be an arbitrary domain of the x-plane, and let $f(x, y)$ be analytic in the points $(x, 0)$, where x lies in T.

Let B be a regular region lying within T, and let p_x be a positive real function of x, such that

(1) $\log p_x$ is harmonic within B:

$$\Delta \log p_x = 0;$$

(2) p_x is continuous on the boundary C of B, and the boundary values $p_x|_C$ are positive.

If, now,

$$R_x|_C \geqq p_x|_C,$$

then, throughout the whole interior of B,

$$R_x \geqq p_x.$$

Finally, if at a single interior point the lower sign holds, then $R_x = p_x$ throughout B.

Hartogs finds further that if R_x is continuous together with its partial derivatives of the first and second orders, then R_x satisfies the differential inequality:

$$\Delta \log R_x \leqq 0.$$

The definition and the properties of the function R_x here considered have been extended to the case that $f(x, y)$ is allowed to be meromorphic instead of being restricted to being analytic, in a paper by Levi.†

* *Math. Ann.*, 62 (1905), p. 46.

† E. E. Levi, *Ann. di mat.* (3), 17 (1910), p. 12.

In addition, Cousin establishes the general existence theorem for this case, namely, that the zeros and the singularities may be chosen at pleasure. More precisely, this condition is as follows. To each point $(a) = (a_1, \cdots, a_n)$ of finite space shall be assigned a definite region $T_{(a)}$ including this point in its interior, and a function

$$f_{(a)}(z_1, \cdots, z_n) = \frac{H_{(a)}(z_1, \cdots, z_n)}{G_{(a)}(z_1, \cdots, z_n)},$$

where $G_{(a)}$ and $H_{(a)}$ are both analytic in $T_{(a)}$ and where, in case both functions vanish at the same point of $T_{(a)}$, they have no common factor there.* When two regions $T_{(a)}$ and $T_{(b)}$ overlap, the corresponding functions $f_{(a)}(z_1, \cdots, z_n)$ and $f_{(b)}(z_1, \cdots, z_n)$ shall be *equivalent* in the common region, i. e., their quotient, taken either way, shall remain finite, and so shall have at most removable singularities there.

Under these hypotheses there exist two integral functions, $G(z_1, \cdots, z_n)$, $H(z_1, \cdots, z_n)$, such that their quotient

$$f(z_1, \cdots, z_n) = \frac{H(z_1, \cdots, z_n)}{G(z_1, \cdots, z_n)}$$

is equivalent to $f_{(a)}(z_1, \cdots, z_n)$ in the region $T_{(a)}$ for all values of (a) and that, at all points at which G vanishes, this quotient is in normal form.

From the theorems of the next paragraph it appears that both numerator and denominator can be written as the (finite or infinite) product of prime factors.

But Cousin's methods extend far beyond the scope of this case. Cousin states the following theorem.† Let $S = (S_1, \cdots, S_n)$ be an arbitrary cylindrical region. To each interior point $(a) = (a_1, \cdots, a_n)$ let a region $T_{(a)}$ lying in S and including (a) in its interior, and a function $f_{(a)}(z_1, \cdots, z_n)$ analytic in (a) be given. When two regions $T_{(a)}$ and $T_{(b)}$ overlap, the corre-

* Cf. IV, § 1. The denominator function $G_{(a)}(z_1, \cdots, z_n)$ will, of course, in general not vanish at all, and in that case can be set $= 1$.

† L. c., p. 60, Theorems XIII, XIV.

§ 10. The Representation of Certain Meromorphic Functions as Quotients

In his noted memoir of 1876 Weierstrass showed that any function of a single complex variable, which has no other singularities than poles in the finite region of the plane, can be expressed as the quotient of two integral (rational or transcendental) functions.

The theorem was later extended by Mittag-Leffler to the case of an arbitrary region. A function meromorphic in such a region can be expressed as the quotient of two functions each analytic in the region. In both cases, the numerator function and the denominator function never vanish at the same point of the region.

Furthermore, the region may be any continuum whatever, and both the zeros and the poles may be chosen arbitrarily in it. There will always exist a function with the given zeros and poles and otherwise analytic and different from zero in the given region.

The first of these theorems admits generalization for a function of several variables. If $f(z_1, \cdots, z_n)$ is meromorphic at every point of finite space, then there exist two (rational or transcendental) integral functions $G(z_1, \cdots, z_n)$ and $H(z_1, \cdots, z_n)$ such that

$$f(z_1, \cdots, z_n) = \frac{H(z_1, \cdots, z_n)}{G(z_1, \cdots, z_n)}.$$

Moreover, at any point at which G and H both vanish, the representation is a normal one; i. e., G and H have no common factor in this point; IV, § 1.

This theorem was stated by Poincaré for the case of two variables, and he gave a proof based on harmonic functions in four-dimensional space.* A more elementary proof, which applies, moreover, to the general case of n variables, was later published by Cousin.†

* *Acta Math.*, 2 (1883), p. 97.

† *Acta Math.*, 19 (1895), p. 1.

write it as the product of two integral functions:

$$G(z_1, \cdots, z_n) = G_1(z_1, \cdots, z_n)G_2(z_1, \cdots, z_n),$$

both of which vanish.*

The roots of a prime function yield the coordinates of all finite points of a certain monogenic analytic configuration.

Let $G(z_1, \cdots, z_n)$ vanish in a point, but not vanish identically. Then the equation

$$G(z_1, \cdots, z_n) = 0$$

defines one or more monogenic analytic configurations. Let M denote one of them. By the aid of Cousin's theorem it is possible to infer the existence of an integral function which vanishes in the points of M and nowhere else, and which, moreover, is prime.† $G(z_1, \cdots, z_n)$ is divisible by this function.

From Weierstrass's factor theorem, IV, § 1, it now follows that, in the neighborhood of a point and hence throughout any finite region of $2n$-dimensional space, an integral function which vanishes there, but does not vanish identically, can be written as the product of a finite number of factors, each irreducible in the point or in the region, multiplied by another integral function which does not vanish there.

It is now an easy matter, by the methods used in the proofs of Weierstrass's and Mittag-Leffler's theorems, to establish the proposed generalization: An integral function which vanishes, but does not vanish identically, can be written in one, and essentially in only one, way as the (finite or infinite) product of its prime factors.

Moreover, the existence theorem for such functions, whose prime factors are arbitrary, holds there. Let $G_1, G_2, \cdots$ be an infinite set of prime functions subject merely to the condition that at no point of finite space do the monogenic analytic configurations which correspond to their roots have a cluster point.

* Gronwall, Thesis, Upsala, 1898, p. 7.

† This theorem is due to Gronwall, l. c. It was rediscovered by Hahn, *Monatshefte*, 16 (1905), p. 29.

sponding functions shall be equivalent in the common domain. Then there exists a function $f(z_1, \cdots, z_n)$ analytic in S and equivalent to $f_{(a)}(z_1, \cdots, z_n)$ in $T_{(a)}$ for all points (a) of S.

This theorem carries with it the other one, in which the word *analytic* is replaced by *meromorphic*, and, in the conclusion the function f is expressed as a quotient:

$$f(z_1, \cdots, z_n) = \frac{H(z_1, \cdots, z_n)}{G(z_1, \cdots, z_n)},$$

which, at any point (a) in which both numerator and denominator vanish, is in reduced form.

Dr. Gronwall* has just shown by an example that the theorem in this degree of generality is not true. It is true if the region S is simply connected, i. e., if each region S_k is simply connected. One of these regions, S_k, however, may be multiply connected.

§ 11. Integral Functions as Products of Prime Factors

A further theorem which Weierstrass established in the memoir of 1876 is this. If $G(z)$ be any integral function which does not vanish identically, but which has an infinite number of roots, then $G(z)$ can be written as an infinite product of prime functions:

$$G(z) = \Gamma(z)\, z^m \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{g_n(z)},$$

where

$$g_n(z) = \frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n},$$

and $\Gamma(z)$ is an integral function having no roots.

As has already been pointed out, the zeros may be chosen at pleasure.

To extend this theorem to integral functions of several complex variables it is necessary first of all to define a prime function.

An integral function $G(z_1, \cdots, z_n)$, which vanishes for some point, is said to be *prime* or *irreducible* if it is not possible to

* *Bull. Amer. Math. Soc.* (2) 20 (1914), p. 173.

of the variables (x, y) not being the whole finite space, but an arbitrary cylindrical space. His proofs are based on Cousin's theorem, concerning which we have reported at length, and his conclusions are, therefore, restricted in the same measure as Cousin's theorem is restricted. Thus Gronwall's example would vitiate some of Hahn's theorems in the generality in which Hahn stated them.

Then there exists an integral function whose roots are those of $G_1, G_2, \cdots$ and which include no other points.*

This theorem suggests the question, what are the characteristic properties of a monogenic analytic configuration M, that its finite points may be identical with the roots of an integral function,

$$(1) \qquad G(z_1, \cdots, z_n) \;?$$

First, as regards the function G, it is clear that this must be irreducible, or a power of an irreducible function.

Next, let $(a_1, \cdots, a_n)$ be any point of finite space which is a cluster point of points of M. A necessary condition that M be given by (1) is seen to be that $G(a_1, \cdots, a_n) = 0$. Hence all the points of M that lie in the neighborhood of $(a_1, \cdots, a_n)$, and no others, will be given by the vanishing of a finite number of functions, $G_k(z_1, \cdots, z_n)$, each analytic at $(a_1, \cdots, a_n)$ and vanishing there, and each irreducible there.

Conversely, this condition is sufficient. More precisely, let M be a monogenic analytic configuration of the $(n - 1)$st grade ($= (n - 1)$-ter Stufe) in the domain of the n variables $(z_1, \cdots, z_n)$, and let it be such that, if $(a_1, \cdots, a_n)$ be any finite cluster point of points of M, then the points of M which lie in the neighborhood of $(a_1, \cdots, a_n)$ are given by a finite number of equations, $G_k(z_1, \cdots, z_n) = 0$, where each of these functions is analytic in the point $(a_1, \cdots, a_n)$ and vanishes there, and moreover is irreducible there. Then the finite points of M are coincident with the roots of an irreducible integral function $G(z_1, \cdots, z_n)$.

This theorem was stated and proven by Hahn† for the case $n = 2$, the formulation there being slightly different. Hahn also states more general theorems, the four-dimensional space

* Appell, *Acta Math.*, 2 (1883) p. 71. Biermann, *Sitzungsber. der Wiener Akad.*, 89 (1884), 2. Abteil., p. 266. Biermann also considers the generalization of Mittag-Leffler's theorem in its more restricted form to functions of several variables. Certain wider forms of the theorems can be treated in the same manner.

† L. c.

a single boundary point of T. This theorem has, moreover, recently been extended to the most general Riemann's surface.*

It is clear from the foregoing that such a theorem cannot hold for functions of more than one variable.

§ 2. Non-Essential Singularities

The analogue of a pole of a function of a single variable is a point $(a_1, \cdots, a_n)$, in whose neighborhood the function can be written in the form

$$F(z_1, \cdots, z_n) = \frac{H(z_1, \cdots, z_n)}{G(z_1, \cdots, z_n)}, \tag{1}$$

where G and H are both analytic at $(a_1, \cdots, a_n)$, and

$$G(a_1, \cdots, a_n) = 0, \quad H(a_1, \cdots, a_n) \neq 0.$$

Here, F becomes infinite for all methods of approach to the point, just as in the case $n = 1$. We shall denote such a point as a *pole*, or as a *non-essentially singular point of the first kind*.†

But even a rational function can have a more complicated singularity. Suppose that G and H are polynomials relatively prime to each other, both vanishing at $(a_1, \cdots, a_n)$; e. g.,

$$F(w, z) = \frac{w}{z}, \qquad (a_1, a_2) = (0, 0).$$

Here, the function can actually take on any arbitrarily assigned value in a point of an arbitrarily assigned neighborhood of the singular point in question.

We are led, then, to a second kind of singularity, the function still being of the form (1), but H vanishing also at the point in question, though still being prime to G. Such a point is called a *non-essentially singular point of the second kind.* In the neighborhood of such a point, which we will take as lying at the

* The question has been treated by Koebe, Freundlich, and Osgood; cf. Osgood, Funktionentheorie, v. 1, 2d ed., 1912, p. 747.

† Weierstrass, Werke, 2, p. 156.

LECTURE III

SINGULAR POINTS AND ANALYTIC CONTINUATION

§ 1. Introduction

The simplest singular points which an analytic function of a single complex variable can have are poles, isolated essential singularities, and branch-points.

A function of several complex variables cannot have an isolated singularity, if we except the trivial case of a removable singularity, i. e., a singularity such that the function becomes analytic at the point in question when a suitable value is assigned to it there.*

For example, the function of the single variable z,

$$f(z) = \frac{1}{z}$$

has an isolated singularity at the point $z = 0$.

But the function of the two complex variables $w = u + vi$, $z = x + yi$:

$$F(w, z) = \frac{1}{z},$$

has a whole two-dimensional manifold of singularities in the four-dimensional space of these variables, namely, the points $(u, v, 0, 0)$.

It is a theorem due to Weierstrass and proven by Runge† that to an arbitrary continuum T of the complex z-plane there correspond functions of z which are analytic at every point of T and which furthermore cannot be continued analytically over

* This result can be obtained directly from Cauchy's integral formula or Laurent's series. It was stated by Hurwitz in his Zürich address, *Verh. des I. intern. Math.-Kongresses*, 1897, p. 104.

† *Acta*, 6 (1884), p. 229.

A function which has no other singularities in a given region or in the neighborhood of a given point than non-essential ones is said to be *meromorphic* in the region or in the point.

§ 3. Essential Singularities

An analytic function of a single complex variable z may have an isolated essential singularity, $z = a$, of either one of two kinds: (a) the function may be analytic throughout the complete neighborhood of the point a except at the point itself, and there neither remain finite nor become infinite; (b) the function may have poles that cluster about the point a, being analytic at all other points of the neighborhood distinct from a.

It follows from the first theorem of § 1 that the first case has nothing corresponding to it when we pass to functions of several complex variables. But may not the second case be realized? May not a function of several variables, $f(z_1, \cdots, z_n)$, be analytic except for non-essential singularities throughout the whole neighborhood of a point $(a_1, \cdots, a_n)$, this point alone being excepted? Weierstrass believed apparently that it can, for he stated the following theorem.*

To an arbitrary continuum in the $2n$-dimensional space of the variables $(z_1, \cdots, z_n)$ there correspond functions analytic or having at most non-essential singularities, but having in every boundary point a singularity of higher order.

This theorem, however, is false, as was shown by E. E. Levi† in a notable paper published three years ago, to which we shall return later, §§ 8, 9. In particular, it appears that an isolated essential singularity is impossible.

§ 4. Removable Singularities

In his inaugural dissertation Riemann‡ stated and proved the theorem whose practical value is so well known, namely, that

* *Journ. für Math.*, 89 (1880), p. 5 = Werke, 2, p. 129.

† *Ann. di. mat.* (3), 17 (1910), p. 61. Levi's paper appeared while a paper of Hartogs, *Math. Ann.*, 70 (1911), p. 217, overlapping to some extent Levi's paper and showing in particular the impossibility of an isolated essential singularity, was in press.

‡ Göttingen Dissertation, 1851, § 12, = Werke, p. 23.

origin, $(0, \cdots, 0)$, the function can in general* be written in the form:

$$(2)\quad F(z_1, \cdots, z_n) = \frac{z_n^m + A_1 z_n^{m-1} + \cdots + A_m}{z_n^l + B_1 z_n^{l-1} + \cdots + B_l}\,\Omega(z_1, \cdots, z_n),$$

where the coefficients A, B are functions of $(z_1, \cdots, z_{n-1})$, each analytic at the point $(0, \cdots, 0)$ and vanishing there, the two polynomials in z_n being prime to each other; and where, moreover, Ω is analytic and not zero at the origin.

In every neighborhood of a pole there are other poles, their locus being the $(2n - 2)$-dimensional analytic manifold or manifolds

$$G(z_1, \cdots, z_n) = 0.$$

But there are no other singularities in the neighborhood in question. For the special case $n = 2$ the non-essential singularities of the second kind are isolated points, since two functions $G(w, z)$, $H(w, z)$ which are prime to each other, like two polynomials having this property, can vanish simultaneously only in isolated points. But when $n > 2$, there will be a whole $(2n - 4)$-dimensional locus of singularities of the second kind,—this locus consisting of a finite number of analytic configurations, each of the dimension in question. In fact, the necessary and sufficient condition that the numerator and the denominator of the fraction in (2) vanish simultaneously is that their resultant vanish. The latter is analytic in $z_1, \cdots, z_{n-1}$ and vanishes at the origin; but it does not vanish identically.

As regards the poles which lie in the neighborhood of a singularity of the second kind, they are situated on the manifold, or manifolds,

$$G(z_1, \cdots, z_n) = 0,$$

and they consist of the totality of such points with the exception of those for which H also vanishes, i. e., the singularities of the second kind.

* In any case, a suitable homogeneous linear transformation of $z_1, \cdots, z_n$ will yield a new function for which the statement is true; cf. IV, § 1. The theorems of the paragraph just cited are assumed in the present paragraph.

A special case of this theorem was familiar to Weierstrass, namely, that in which the function f can be written, in the neighborhood of the point in question, as the quotient of two functions, each analytic and vanishing there; cf. I, § 1, and IV, § 1.

This latter theorem of Kistler's admits an extension. The excepted points may fill a $(2n - 3)$-dimensional manifold, the latter being such that, if we set $z_k = x_k + i y_k$, then three of the $2n$ coordinates, as y_{n-1}, x_n, y_n, can be expressed as single-valued or finitely multiple-valued continuous functions of the remainder in the neighborhood in question. And cases reducible to the latter by linear transformation of the complex variables are obviously included.

Related to these theorems more or less closely is a further theorem stated by Kistler,* but not proven by him. From the neighborhood T of a point $(a_1, \cdots, a_n)$ let the points of a set L be excluded, L consisting of the points of a finite number of analytic manifolds, each of dimension $2n - 4$ or lower; and let the remainder of T be denoted by T'. In the region T' $f(z_1, \cdots, z_n)$ shall be meromorphic. Then the function can have in the points of L no higher singularities than removable and non-essential ones.

The proof of this theorem was later given by Hartogs.†

We note, however, in closing this paragraph an interesting application which Kistler makes of the latter theorem to a proof of Jacobi's theorem of inversion, I, § 2.

§ 5. Analytic Continuation by Means of Cauchy's Integral Formula

During the last few years a number of important theorems on analytic continuation have been discovered, chiefly through the

* L. c. In the light of Gronwall's recent discovery concerning the scope of Cousin's theorem, Kistler's proof was even more restricted than appeared at the time.

† *Math. Ann.*, 70 (1911), p. 217. Kistler appears to have had no substantial reason for supposing the theorem to be true, for his proof is based on a misunderstanding of Cousin's results, II, § 9. The chief credit for the theorem would seem, therefore, to be due to Hartogs.

if a function $f(z)$ is analytic throughout the neighborhood T of a point $z = a$ with the possible exception of this point itself, and if $f(z)$ remains finite in T, then $f(z)$ approaches a limit when z approaches a; and if the function is defined for $z = a$ as equal to its limiting value there, then it is analytic in this point also.

This theorem admits a number of generalizations or extensions for functions of several variables. The most obvious one was stated and proven by Kistler* in the following formulation. Let $f(z_1, \cdots, z_n)$ be analytic throughout a region T consisting of the neighborhood of a point $(a_1, \cdots, a_n)$ with the exception at most of the points of a $(2n - 2)$-dimensional analytic manifold L; and let the function remain finite in T. Then the function will approach a limit in the points of L and will be analytic there if suitably defined there.

Similarly, Riemann's theorem, that a function $f(z)$ which is analytic in a region S except along a simple regular curve C, where it is continuous, is also analytic in the points of C, can be generalized. If $f(z_1, \cdots, z_n)$ be analytic in a $2n$-dimensional region except in the points of a single $(2n - 1)$-dimensional analytic† manifold $\mathfrak{C}$, where f is continuous, then f is analytic in the points of $\mathfrak{C}$ also. This theorem is not mentioned by Kistler.

A second generalization was given by Kistler, and is as follows. Let $f(z_1, \cdots, z_n)$ be analytic throughout the neighborhood of a point $(a_1, \cdots, a_n)$ with the exception at most of the points of a finite number of analytic manifolds, each of which is at most $(2n - 4)$-dimensional. No hypothesis, however, is now made regarding the function's remaining finite. Such a function will be analytic in the excepted points also, if properly defined there. For $n = 2$, this becomes the first theorem of § 1.

* Göttingen dissertation, Ueber Funktionen von mehreren komplexen Veränderlichen, § 7, Basel, 1905. This theorem, like the original theorem of Riemann's, is exceedingly serviceable in practice, and was probably used before Kistler's enunciation and proof of it. An important special case was familiar to Weierstrass; I, § 1, end. A second proof is contained substantially in Hartogs's paper, *Math. Ann.*, 70 (1911), p. 217.

† More general manifolds are also admissible.

boundary of (B, B') consisting of the closed surface described in the condition (c) of the theorem.

This double integral, however, represents a function which is analytic in the two independent variables (x, y) throughout the whole interior of (B, B'), and which, furthermore, coincides with the given function throughout the interior of (B, K). Hence the proposition is established.

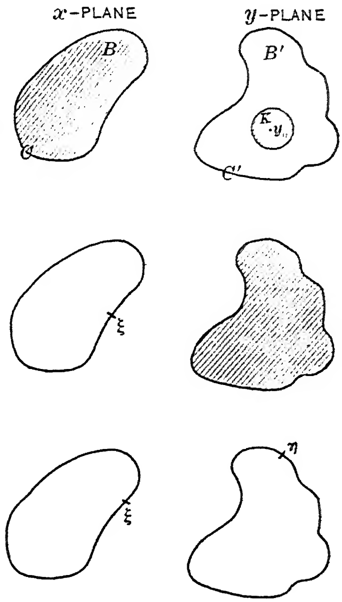

FIG. 1.

It is interesting to notice the nature of the hypotheses. (a) imposes a condition on the function in each point of a 4-dimensional region; (b) is 3-dimensional, in that it is made up of a 1-dimensional system of 2-dimensional hypotheses; while (c) is 2-dimensional.

Again, the points of (a) form a 4-dimensional piece of the 4-dimensional cylindrical region (B, B'). The points of (b) form one or more pieces of the 3-dimensional boundary of (B, B'). This latter manifold, it will be remembered, consists of but a

researches of Hartogs and E. E. Levi. We begin with the former.

*Hartogs's Theorem.** Let B, B' be regular regions of the x-plane and y-plane respectively, and let K be the neighborhood of an interior point y_0 of B'. Let $f(x, y)$ be a function with the following properties, cf. Fig. 1:

(*a*) In the interior of the four-dimensional cylindrical region (B, K), $f(x, y)$ shall be analytic; and, moreover, for every point y' of K, $f(x, y')$, regarded as a function of x alone, shall be continuous on the boundary C of B.

(*b*) For every point ξ of C, $f(\xi, y)$, regarded as a function of y alone, shall be analytic within B' and continuous on the boundary C' of B'.

(*c*) In that part of the boundary of (B, B') which is determined by the points (ξ, η) where ξ ranges over C and η over C', $f(\xi, \eta)$ shall be a continuous function of (ξ, η).

Then $f(x, y)$ can be continued analytically throughout the interior of the entire cylindrical region (B, B').

The proof of this theorem is simple. For every interior point (x, y) of (B, K), $f(x, y)$ can be represented by Cauchy's integral formula:

$$f(x, y) = \frac{1}{2\pi i}\int_C \frac{f(\xi, y)}{\xi - x}\, d\xi.$$

Again, by Cauchy's integral formula,

$$f(\xi, y) = \frac{1}{2\pi i}\int_{C'} \frac{f(\xi, \eta)}{\eta - y}\, d\eta.$$

Hence

$$f(x, y) = \frac{1}{(2\pi i)^2}\int_C \frac{d\xi}{\xi - x}\int_{C'} \frac{f(\xi, \eta)}{\eta - y}\, d\eta$$

$$= \frac{1}{(2\pi i)^2}\int_S\!\!\int \frac{f(\xi, \eta)}{(\xi - x)(\eta - y)}\, d\xi\, d\eta,$$

where the double integral is extended over the part S of the

* *Sitzungsber. der Münchener Akad.*, 36 (1906), p. 223. The formulation here given is slightly different from that of Hartogs.

(*a*) in every point $(x_1, \cdots, x_k, a_{k+1}, \cdots, a_n)$, where x_i, $i = 1, \cdots, k$, ranges over B_i, and a_j, $j = k+1, \cdots, n$, is a fixed point in B_j;

(*b*) in every point $(\xi_1, \cdots, \xi_k, x_{k+1}, \cdots, x_n)$, where ξ_i, $i = 1, \cdots, k$, ranges over the boundary C_i of B_i and x_j, $j = k+1, \cdots, n$, ranges over B_j.

The function will then admit analytic continuation throughout the cylindrical region $(B_1, \cdots, B_n)$.

In the foregoing results is contained the remarkable theorem that a function $f(x_1, \cdots, x_n)$ which is analytic in every boundary point of a cylindrical region $(B_1, \cdots, B_n)$ admits analytic continuation throughout the whole region.*

This theorem holds for the general case of any four-dimensional region, whether cylindrical or not. Cf. § 9.

§ 6. Application to the Distribution of Singularities

From the main theorem of the last paragraph Hartogs deduces the following theorem relating to the distribution of the singularities of an analytic function.

Theorem. Let $f(x, y)$ be analytic in the points $(0, y)$, where $0 < |y| < h$, and let f have a singular point at the origin, $(0, 0)$. Then, to each point x' of a certain region B: $|x| < \rho$, will correspond at least one point y' of the region B': $|y| < h$, such that $f(x, y)$ has a singular point in (x', y').

Here, again, it is useful to picture the points to ourselves in the plane of analytic geometry. We assume the function $f(x, y)$ to be analytic along that part of the y-axis which lies in the neighborhood of the origin, this latter point alone being excepted and the function being in fact singular there. The conclusion is that the projections on the x-axis of the singular points of the function which lie in the rectangle (B, B') completely cover that part of the axis which lies in this rectangle.

We must not, of course, think of the singular points as dividing the part of the region (B, B') in which the function is considered,

* Hartogs, l. c., p. 231.

single piece. Finally, the points of (c) yield one or more 2-dimensional pieces of the 3-dimensional boundary of (B, B') just referred to, and they also lie in the points of (b).

A further aid toward a geometric realization of the hypotheses is obtained if we picture the cylindrical region (B, B') as a rectangle in the plane of analytic geometry. Here, as in the use of that plane in the study of plane curves when the complex points are admitted to the discussion, we have, it is true, only a two-dimensional figure for a four-dimensional set of geometric objects; and we have to work by analogy.

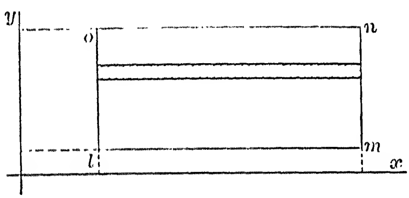

FIG. 2.

Condition (a) is now seen to refer to the points of a narrow strip that courses the large rectangle $lmno$, the latter representing the region (B, B'). Conditions (b) and (c) have to do merely with points of the boundary, which lie in the sides lo and mn. In the conclusion, the function is extended over an enlarged region dimensionally coordinate with the slender strip of condition (a).

The extension of the theorem in the above formulation to the case of n-variables is obvious.

For three variables, the geometric interpretation last considered leads to a rectangular parallelepiped, coursed by a slender one with parallel faces, and the further conditions of the theorem are interpretable in terms of regions and curves lying in the faces of the large parallelepiped.

Another form of the hypotheses of the theorem, somewhat less general, but more compact, consists in requiring the function $f(x_1, \cdots, x_n)$ to be analytic

§ 7. Generalizations of the Theorem of § 5

Hartogs has given substantially the following generalization of the theorem of § 5.

Theorem. Let B and B' be regular regions of the x- and y-planes respectively. Let $f(x, y)$ be a function with the following properties.

(*a*) $f(x, y)$ shall be analytic in every point (ξ, y), where ξ ranges over the complete boundary of B and y ranges over the interior and boundary of B'.

(*b*) $f(x, y)$ shall be analytic in every point $(x, \psi(x))$, where $\psi(x)$ is analytic in every interior and boundary point of B, and where, moreover, the points $y = \psi(x)$ lie within B'.

(*c*) B' shall be simply connected. More generally, the points of B' which correspond to the points of B through the function $y = \psi(x)$ shall be capable of being enclosed in a simply connected region lying in B'.

Then the function $f(x, y)$ can be continued analytically throughout the whole 4-dimensional region (B, B').

The geometric picture of the conditions by means of figures in the (x, y)-plane is here a distinct aid. The hypotheses relate to a narrow 4-dimensional region which encloses part, but not all, of the boundary of (B, B'), and which, furthermore, penetrates

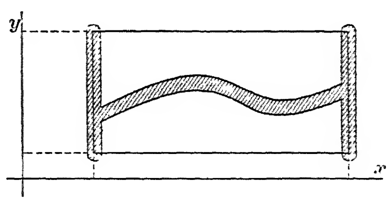

Fig. 3.

into the interior of (B, B'). They are suggestively indicated by the accompanying figure. In one minor respect this representation is defective, since the part of the boundary of (B, B') to which the hypotheses relate does not necessarily consist of more than a single piece.

in two. In this respect, the geometric analogy in the plane is defective.

Levi has given a similar theorem for the case that $f(x, y)$ is allowed to be meromorphic instead of being restricted to being analytic; cf. § 8, Lemma 2. The formulation of that lemma affords a more precise statement for Hartogs's theorem.

Continuation. Singular Surfaces. The theorem of the preceding paragraph dealt with functions $f(x, y)$ which have *at least* one singular point (x, y) in the neighborhood of the origin $(0, 0)$ corresponding to every x near $x = 0$. We turn now to a theorem which has to do with functions which have a two-dimensional assemblage of singular points spread out over a surface.

*Theorem.** Let

$$y = \varphi(x)$$

be a single-valued continuous function of the complex variable x defined throughout a certain neighborhood of the point $x = 0$. Let $f(x, y)$ be analytic at all points of the neighborhood of the origin, $(0, 0)$, with the exception of the points $(x, \varphi(x))$, and let these be singular points of $f(x, y)$ which are not removable singularities. Then $\varphi(x)$ is an analytic function of x.

The point of this theorem is the very great restriction to which the singularities of an analytic function of several complex variables are seen to be subject. When we consider that the singular points of an analytic function of a single complex variable are as arbitrary as the boundary of a 2-dimensional continuum, the essential change in the situation on passing to functions of several variables becomes evident.

To this subject belongs a theorem of Levi's, to which we shall turn in § 10.

In the same paper, Hartogs has generalized this latter theorem so that it applies to functions of any number of variables, and these functions may be multiple-valued.

* Hartogs, *Acta*, 32 (1908), p. 57. The proof of this theorem is complex, and is based on a series of earlier developments.

That the condition is sufficient appears at once on multiplying the equation (1) through by the function

$$A_0(x)\,y^l + A_1(x)\,y^{l-1} + \cdots + A_l(x)$$

and writing the right-hand side as a new Laurent series in y. The negative powers of y are seen to disappear, and hence the function represented by the series is analytic in the domain $|x| \leqq h$, $|y| \leqq K$.

The proof that the condition is necessary, though longer, is not complex.

Definition. A function $f(x_1, \cdots, x_n)$ shall be said to be *continued meromorphically* from the point A to the point B along a simple path L of $2n$-dimensional space if the function is meromorphic at A and if, on enclosing L in a slender simply connected* $2n$-dimensional tube, the function, when continued analytically within this tube, presents no other than non-essential singularities there. The resulting function will then be single-valued in those points of the tube in which it is defined.

A function can be continued meromorphically throughout a region if it can be continued meromorphically along every simple path lying in the region.

Lemma 2. Let $f(x, y)$ be meromorphic throughout a certain 4-dimensional continuum T containing the points $(0, y)$ where $0 < |y| \leqq K$, but not containing the origin, $(0, 0)$; and let it not be possible to continue $f(x, y)$ meromorphically to the origin along a path lying, except for its extremity $(0, 0)$, in T.

Then there exists a circle S: $|x| < h$, such that (a) the points (x, y), where x lies in S and $|y| = K$, lie in T, and thus $f(x, y)$ is meromorphic in each of them; (b) to each point x' of S corresponds a point (x', y'), $y' = r'e^{\theta i}$, $r' < K$, such that $f(x, y)$ can be continued meromorphically along the path $x = x'$, $y = re^{\theta i}$ (where r is the independent variable and x', θ are con-

* By a *simply connected* region, or more precisely a *linearly* simply connected region, is meant one such that a simple closed curve lying in it can be drawn together continuously to an interior point of the region without meeting the boundary.

The theorem admits of generalization to functions of any number of variables; Hartogs, l. c.

Levi* has given a similar theorem for the case that $f(x, y)$ is meromorphic in (B, K) and also in the points (ξ, y), where ξ lies on C and y in B'. $f(x, y)$ will then be meromorphic in (B, B').

§ 8. Levi's Memoir of 1910

We come now to one of the most important contributions of recent years to the theory we are discussing,—E. E. Levi's memoir of 1910.* With the aid of two lemmas, each admitting a simple proof, Levi establishes a fundamental theorem, from which follows with ease a complete treatment of a number of questions in our theory which had presented themselves during the last decade.

Lemma 1. Let $f(x, y)$ be analytic in the cylindrical domain

$$B: \qquad |x| \leqq h, \quad k \leqq |y| \leqq K;$$

and let $f(x, y)$ be developed in that domain into a Laurent's series:

$$f(x, y) = \sum_{n=-\infty}^{\infty} g_n(x) y^n. \tag{1}$$

In order that the analytic continuation of $f(x, y)$ into the region

$$|x| \leqq h, \quad |y| < k$$

may have there no other singularities than non-essential ones, it is necessary and sufficient that there exist a system of $l + 1$ functions $A_i(x)$, $i = 0, 1, \cdots, l$,—where l may be any positive integer,—such that

$$A_0(x)\, g_{n-l}(x) + A_1(x)\, g_{n-l+1}(x) + \cdots + A_l(x)\, g_n(x) = 0,$$

where $n = -1, -2, \cdots$. Moreover, the functions $A_i(x)$ are all analytic in the circle $|x| \leqq h$, and $A_0(x)$ does not vanish there.†

* *Ann. di Mat.* (3), 17 (1910), p. 61.

† This latter function may, without loss of generality, be set equal to unity.

morphic, it is possible to enunciate the foregoing theorem for the case that the word *meromorphic* is changed throughout to *analytic,* and moreover *essential singularity* is replaced by *singularity.*

Both the second lemma and the main theorem are formulated here more generally than in Levi's paper. Levi's proof applies, however, to the extended theorems.

§ 9. Continuation. Lacunary Spaces

Regular Regions. We will understand by a *regular region* of m-dimensional space a finite m-dimensional continuum, together with its boundary; the latter consisting of a finite number of simple, regular, closed, non-intersecting, $(m-1)$-dimensional manifolds. It is obvious that this definition can be formulated more generally, but the above is sufficient for our present purposes.

Theorem 1. Let $f(x, y)$ be analytic at every point of the boundary of a regular region, T, of the 4-dimensional space of (x, y). Then $f(x, y)$ admits an analytic continuation throughout T, and the resulting function will be analytic * in T.

This theorem corresponds to Levi's Corollary I, l. c., p. 11, but is more general. His statement of his corollary is defective. We will speak of the proof after taking up the proof of the next theorem.

In particular, then, it follows from the foregoing theorem that an analytic function of two complex variables cannot have a finite lacunary region, around which the function is analytic. Thus, for example, no function $f(x, y)$ exists which is analytic in the spherical shell bounded by the hyperspheres with centres at the origin and of radii, $r = 1$ and $r = 1 + \delta$, $\delta > 0$, and which has in the former hypersphere a natural boundary.

As has already been pointed out, the theorem was stated and proven for cylindrical regions by Hartogs.

Theorem 2. This theorem differs from Theorem 1 solely in

* Cf. II, § 1.

stants, and where, moreover, $r' < r \leqq K$), into the neighborhood of (x', y'), but not to this point.

It will be observed that if the conditions of the theorem are fulfilled for a given K, then they are also fulfilled for any smaller K.

We can picture the loci $x = x' = \alpha$, $|\alpha| < h$, as surfaces which form a *field* (in the sense in which this word is used in the Calculus of Variations) in the 4-dimensional neighborhood of the origin. If we make any analytic transformation of this neighborhood, of the form

$$u = \varphi(x, y), \qquad v = \psi(x, y),$$

where φ and ψ are functions of the complex variables (x, y) analytic at the origin, and where the Jacobian of φ and ψ does not vanish, we can then state an obvious corollary of Lemma 2 for the surfaces in the (u, v)-space, into which the surfaces $x = \alpha$ have been carried.

This is all the preparation Levi needs for his main theorem, to which we now turn.

Theorem. Let E be a perfect set of points in the 4-dimensional space of the complex variables x, y; and let O be a fixed point of this space. Let r be the distance from O to a variable point of E. If there be a point P of E for which r has a relative maximum,* then there cannot exist a function $f(x, y)$ which is meromorphic in the neighborhood of P except for the points of E, and in each of those points has an essential singularity.

More precisely stated, the conclusion is this. Consider the continuum, T, exterior to the hypersphere through P with O as centre and interior to a small hypersphere with P as centre. Then there cannot exist a function meromorphic in T and not admitting meromorphic continuation at P.

Since Hartogs† has proven Lemma 2 for the case that $f(x, y)$ is required to be analytic instead of being allowed to be mero-

* *Maximum* is here to be understood as meaning that r shall not, in the neighborhood of P, take a larger value than at P; but it may attain that value at other points of the neighborhood.

† § 6, first theorem.

§ 10. Concerning the Boundary of the Domain of Definition of $f(x, y)$

Let Σ be a simple regular 3-dimensional manifold of 4-dimensional space. Then Σ can be represented analytically by the equation

$$\varphi(x_1, x_2, y_1, y_2) = 0,$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$; where, furthermore, φ is continuous together with its first partial derivatives; and where, finally, not all of these four derivatives vanish simultaneously. We will restrict ourselves to such manifolds Σ as correspond to functions φ having continuous second derivatives.

Levi raises the question: Can a given manifold of the above description, or a restricted piece of it, serve as part of the boundary of a region in which a function $f(x, y)$ is meromorphic, but beyond which $f(x, y)$ cannot be continued meromorphically across any part of Σ? In other words, can Σ, or a piece of Σ, be a natural boundary?

He finds that the answer is, in general, negative; since, for it to turn out affirmative, φ must satisfy the following necessary condition. Let $\varphi > 0$ on the side of Σ where $f(x, y)$ is to be meromorphic. Denote by $\mathfrak{C}(\varphi)$ the following expression:

$$\mathfrak{C}(\varphi) = \Delta_2'\varphi \cdot \Delta_1''\varphi + \Delta_2''\varphi \cdot \Delta_1'\varphi - 2\left(\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial y_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial\varphi}{\partial y_2}\right)$$
$$\times\left(\frac{\partial^2\varphi}{\partial x_1\partial y_1} - \frac{\partial^2\varphi}{\partial x_2\partial y_2}\right) - 2\left(\frac{\partial\varphi}{\partial x_1}\frac{\partial\varphi}{\partial y_2} - \frac{\partial\varphi}{\partial x_2}\frac{\partial\varphi}{\partial y_1}\right)\left(\frac{\partial^2\varphi}{\partial x_1\partial y_2} - \frac{\partial^2\varphi}{\partial x_2\partial y_1}\right),$$

where

$$\Delta_1'\varphi = \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2, \qquad \Delta_2'\varphi = \frac{\partial^2\varphi}{\partial x_1^{\,2}} + \frac{\partial^2\varphi}{\partial x_2^{\,2}},$$

and where $\Delta_1''\varphi$, $\Delta_2''\varphi$ denote similar expressions in y_1, y_2. Then must $\mathfrak{C}(\varphi) \leqq 0$ in all points of Σ.

If $\varphi < 0$ on the side of Σ where $f(x, y)$ is meromorphic, then must $\mathfrak{C}(\varphi) \geqq 0$ in all points of Σ.

From this result it follows that if there is to exist a function

having the word *analytic* replaced throughout by the word *meromorphic*.

The second theorem can be proven as follows. We may without loss of generality assume that the boundary of T is pierced by an arbitrary ray from the origin at most in a finite number of points. On each ray which enters T there will, then, be a finite number of segments lying in T. Let AB be such a segment, and let B be the extremity more remote from the origin. Continue $f(x, y)$ meromorphically from B toward A. If it is possible to reach A on every segment, the theorem is granted. If not, let Q be the first point on AB that cannot be reached from B.

Thus, when all segments are considered, a set of points Q lying in the finite region T are obtained, and this set is, from its source, necessarily closed. Let P be one of its points whose distance from O is a maximum. Then, in that part of the neighborhood of P which lies outside of the hypersphere through P with its centre at O, $f(x, y)$ is meromorphic. The function must, therefore, by Levi's theorem, § 8, admit a meromorphic continuation at P, and here is a contradiction.

The first theorem can be proven in a similar manner by the aid of Levi's theorem of § 8, stated for functions required to be analytic instead of being allowed to be meromorphic.

It thus appears that an analytic function of two complex variables cannot have a finite lacunary space around which the function is meromorphic.

This latter result is in direct contradiction to Weierstrass's theorem of § 3, and appears to be the earliest proof that that theorem is false. From Lemma 2 it follows, however, immediately that an isolated essential singularity is impossible, and thus a more elementary proof is afforded of the incorrectness of that theorem.

analytic at the point in question and taking on the value $p + qi$ along the surface in question,—all this, at least, in a certain neighborhood of the given point?

He finds the answer to be affirmative and the function w to be uniquely determined, provided the surface is not what he calls a *characteristic surface*, i. e., a surface along which an analytic function of two complex variables, which is not identically zero, vanishes. In the case of a characteristic surface, there will in general be no solution of the problem. Suppose, for example, that the surface is $y = 0$,—and the general case of a characteristic surface is reducible to this case. Then

$$w(x, 0) = p + qi,$$

and it is evident that $p + qi$ must be a function of x analytic at the given point.

If this condition is satisfied, there will be, not a single, but an infinite number of solutions.

From these results follow at once the theorems:

If $f(x, y)$ is analytic at a point and vanishes along a non-characteristic surface through that point, no matter how restricted that surface may be, it vanishes identically.

If $f(x, y)$ and $\varphi(x, y)$ are both analytic at a point and take on the same values along a non-characteristic surface through that point, however restricted that surface may be, they are identically equal to each other.

Levi-Civita extends the foregoing theorems to functions of any number of variables.

There is a theorem of Levi's* bearing on these characteristic surfaces. He shows that any three-dimensional manifold $\varphi = 0$ (§ 10), in every point of which $\mathfrak{C}(\varphi) = 0$, is composed of a one-parameter family of characteristic surfaces.

The theorems of these last two lectures have brought out clearly the fact that the analytic functions of several complex variables

* *Ann. di Mat.* (3), 17 (1910), p. 89.

$f_1(x, y)$ meromorphic on one side of Σ and having Σ as a natural boundary; and also a function $f_2(x, y)$ meromorphic on the other side of Σ and also having Σ as a natural boundary, then must

$$\mathfrak{C}(\varphi) = 0.$$

How far are these conditions sufficient? In the present memoir Levi shows that this last condition is sufficient; namely: If $\mathfrak{C}(\varphi) = 0$ in every point of Σ: $\varphi = 0$, then there are functions analytic on each side of Σ, but having Σ as a natural boundary;—all this, at least, when Σ is suitably restricted in extent.

In a later paper Levi* obtains the further result, that if $\mathfrak{C}(\varphi) < 0$ in all points of Σ: $\varphi = 0$, then there exists a function $f(x, y)$ analytic on the side of Σ where $\varphi > 0$ and having Σ as a natural boundary;—all this, at least, when Σ is suitably restricted in extent.

§ 11. A Theorem Relating to Characteristic Surfaces

An analytic surface in space of four dimensions may be represented by a pair of equations:

$$(1) \qquad u(x_1, x_2, y_1, y_2) = 0, \quad v(x_1, x_2, y_1, y_2) = 0,$$

where u and v are real functions of the four real variables, analytic at the point in question, their Jacobian with respect to two of the variables,—say y_1, y_2,—not vanishing there.

Levi-Civita† raises the following question. Suppose two real functions, p and q, are given along such a surface, and are analytic there. Thus p and q may be any functions of x_1, x_2 analytic at the point in question, if these are the preferred variables. Does a function of the complex variables exist:

$$w(x, y), \quad x = x_1 + ix_2, \quad y = y_1 + iy_2,$$

* *Ann. di Mat.* (3), 18 (1911), p. 69.

† *Rendiconti Accad. Lincei* (5), 14 (1905), p. 492. He prefaces his problem by recalling the Cauchy problem for two independent variables, x and y, and an analytic curve C in their plane; an arbitrary sequence of analytic values being assumed along C.

LECTURE IV

IMPLICIT FUNCTIONS

§ 1. Weierstrass's Theorem of Factorization

The following theorem is due to Weierstrass.*

Theorem of Factorization. Let $F(u; x_1, \cdots, x_n)$ be a function of the $n + 1$ variables $u, x_1, \cdots, x_n$, analytic in the origin $(0; 0, \cdots, 0)$ and vanishing there. Let

$$(1) \qquad F(u; 0, \cdots, 0) \not\equiv 0.$$

Then, throughout a certain neighborhood of the origin,

$$T: \qquad |u| < h, \quad |x_k| < h', \qquad k = 1, \cdots, n,$$

the following equation holds:

$$(2) \quad F(u; x_1, \cdots, x_n) = [u^m + A_1 u^{m-1} + \cdots + A_m]\,\Omega(u; x_1, \cdots, x_n),$$

where A_i is analytic in $x_1, \cdots, x_n$ throughout the region $|x_k| < h'$ and vanishes at the origin, and Ω is analytic in $u, x_1, \cdots, x_n$ throughout T and does not vanish there.

If $f(z_0, z_1, \cdots, z_n)$ is any function of $z_0, z_1, \cdots, z_n$, analytic at the origin and vanishing there, but not vanishing identically, it is possible by means of a suitable linear transformation of the $n + 1$ variables $z_0, z_1, \cdots, z_n$ to carry f over into a function $F(u; x_1, \cdots, x_n)$ satisfying the foregoing conditions.

Irreducible Factors. On the theorem of factorization can be based a theory of irreducible factors of an analytic function analogous to the theory in the case of polynomials.† First, as regards division. If $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ are both analytic in the point $(a) = (a_1, \cdots, a_n)$ and Φ does not vanish

* Lithographed, Berlin, 1879; Funktionenlehre, 1886, p. 105 = Werke 2, p. 135. In a foot note of the page last cited Weierstrass says that he has repeatedly given the theorem in his university lectures, beginning with 1860.

† Weierstrass, l. c.

are far less capable of adapting themselves to a preassigned region of definition than is the case with the functions of a single variable.

An explanation is, very likely, to be found in the following fact, to which we have already called attention (II, § 1). The real part of an analytic function of a single variable has to satisfy but a single linear partial differential equation (Laplace's equation). In the case, however, of an analytic function of several variables, the real part has to satisfy a simultaneous system of such equations.

It is not the individual root, but the monogenic analytic configurations which are made up of the roots and which exhaust the latter, that are the analogue of the roots of a function of a single variable. And now it is seen from the factor theorem that the number of such configurations which course the neighborhood of a given root is finite.

Earlier Sources. As appears from the applications already considered, there are two wholly distinct classes of theorems at issue. The theorem of factorization asserts the existence of an identity in $n+1$ independent complex variables, the left-hand side being a function $F(u;\, x_1, \cdots, x_n)$ vanishing at the origin, but such that $F(u;\, 0, \cdots, 0) \not\equiv 0$; and the right-hand side being the product of the two factors described in detail in the statement of the theorem. This theorem is universally admitted to be due to Weierstrass.

On the other hand, such a function put equal to 0:

$$F(u;\, x_1, \cdots, x_n) = 0,$$

defines an implicit function of n arguments. That the latter function is given as the root of a polynomial:

$$u^m + A_1 u^{m-1} + \cdots + A_m = 0,$$

where the A_k's are all analytic in $x_1, \cdots, x_n$ at the point in question and vanish there, follows, it is true, from Weierstrass's theorem. But Weierstrass was not the sole discoverer of this theorem. The theorem is contained substantially in Cauchy's Turin memoir of 1831.* In that paper, Cauchy showed that, to each point $(x_1, \cdots, x_n)$ lying in a certain neighborhood of the point $(a_1, \cdots, a_n)$ in question, correspond precisely m roots of the equation

$$F(u;\, x_1, \cdots, x_n) = 0.$$

Furthermore, if $\Phi(u)$ be any function of u analytic at the point $u = 0$ and vanishing there, and if the above m roots be denoted by $u_1, \cdots, u_m$, then the symmetric function

$$\Phi(u_1) + \cdots + \Phi(u_m)$$

* Cf. Exercices d'analyse, 2 (1841), p. 65.

identically, but does vanish at (a); and if, in the neighorhood of (a), a relation of the form

$$F(z_1, \cdots, z_n) = Q(z_1, \cdots, z_n)\,\Phi(z_1, \cdots, z_n)$$

holds, Q being analytic at (a), then F is said to be divisible by Φ in the point (a). If $G(z_1, \cdots, z_n)$ is analytic in the point $(a) = (a_1, \cdots, a_n)$ and vanishes there, then G is said to be *irreducible* at (a) if no equation of the form exists:

$$G(z_1, \cdots, z_n) = G_1(z_1, \cdots, z_n)\,G_2(z_1, \cdots, z_n),$$

where G_1 and G_2 are both analytic at (a) and both vanish there.

Two irreducible factors are *equivalent* if their quotient, taken either way, presents at most removable singularities.

A function $G(z_1, \cdots, z_n)$ analytic at (a) and vanishing there, but not vanishing identically, can be written in one, and essentially in only one, way as the product of factors each irreducible in (a).

A factor which is irreducible at a given point is not necessarily irreducible at every one of its vanishing points which lies in a certain neighborhood of the point. Hence the expression of a function at a given point as a product of factors each irreducible at that point does not always retain this character when that point is replaced by a second root of the function that lies in the neighborhood of that point.

The theorem of algebraic geometry that two curves or surfaces which have ever so short an arc or small a region in common, must necessarily have a whole irreducible piece in common, finds its counterpart here. Let $F(z_1, \cdots, z_n)$ and $\Phi(z_1, \cdots, z_n)$ both be analytic at the origin and vanish there, and let Φ be irreducible there. If F vanishes at all points in the neighborhood of the origin at which Φ vanishes, then F is divisible by Φ.

The Roots of an Analytic Function of Several Variables. In the case of analytic functions of a single variable the roots are isolated. This theorem appears to be lost for functions of several variables, since such a function which vanishes at all has an infinite number of roots clustering about any given root. The theorem admits, nevertheless, a perfectly good generalization.

Thus we have a form of the factor theorem which holds in all cases and which does not depend on an eventual change of the independent variables by a linear transformation.

A corresponding form for the general case, $n > 1$, would be a valuable contribution, since it is not always feasible, under the conditions of the problem in hand, to make the above linear transformation. The tentative theorem is as follows.

Tentative Theorem. Let $F(u;\, x_1, \cdots, x_n)$ be analytic at the origin and vanish there. Then, throughout a certain neighborhood of the origin,

$$T: \qquad |u| < h, \quad |x_k| < h', \qquad\qquad k = 1, \cdots, n,$$

the following equation holds:

$$F(u;\, x_1, \cdots, x_n) = (A_0 u^m + A_1 u^{m-1} + \cdots + A_m)\,\Omega(u;\, x_1, \cdots, x_n),$$

where A_k, $k = 0, 1, \cdots, n$, is analytic in $x_1, \cdots, x_n$ throughout the region $|x_i| < h'$ and vanishes at the origin when $k > 0$; and where Ω is analytic in T and does not vanish there. A_0 may or may not vanish.

For polynomials the theorem is obvious. I have not succeeded in proving it in the general case except when $n = 1$. But in my attempts at a proof I have seen nothing that discredits the theorem and much that renders it probable. I think the chances are that the theorem is true, and I hope that someone will investigate this question.

§ 3. Algebroid Configurations

Consider the function defined by the equation

$$(1) \qquad F = u^m + A_1 u^{m-1} + \cdots + A_m = 0,$$

where $A_k(x_1, \cdots, x_n)$ is analytic in the point $(x) = (0)$ and vanishes there, and the polynomial is irreducible. Such a function is called an *algebroid function.**

* Poincaré, Thèse, 1879, p. 4. It is sometimes desirable to admit the case that the coefficient of u^m is a function $A_0(x_1, \cdots, x_n)$ analytic at the point $(x) = (0)$ and vanishing there.

is expressed by a definite integral which is seen to represent a function of $x_1, \cdots, x_n$ analytic at the point $(a_1, \cdots, a_n)$ and vanishing there.*

In order, then, to obtain the implicit function theorem it remains merely to set

$$\Phi(u) = u^k, \qquad k = 1, \cdots, m,$$

and then express the elementary symmetric functions by the familiar formulas in terms of the Newtonian sums,

$$s_k = u_1{}^k + \cdots + u_m{}^k.$$

Furthermore, Cauchy applied his method to the solution of a problem in implicit functions, namely, to the development of a function into a series of Lagrange. Thus this noted series, so prominent in the early history of the theory of functions, again makes contact with modern analysis.

There are two other proofs of the implicit function theorem considered above, both of which antedate Weierstrass's publication in the *Funktionenlehre*, namely, Poincaré's and Neumann's.†

§ 2. A Tentative Generalization of the Theorem of Factorization

In the case $n = 1$, in which F depends on only two variables, u, x, it is possible to dispense with the condition (1) altogether, provided $F(u, x)$ does not vanish identically, the relation (2) being then modified as follows:

$$F(u, x) = x^l (u^m + A_1 u^{m-1} + \cdots + A_m)\, \Omega(u, x),$$

where l is a positive integer, or 0. Even the proviso just mentioned can be avoided if we write

$$F(u, x) = (A_0 u^m + A_1 u^{m-1} + \cdots + A_m)\, \Omega(u, x).$$

* The work is carried through for the case $n = 1$, our function $F(u; x_1, \cdots, x_n)$ being represented there by $f(x, y)$ and the above function $\Phi(u)$ by $F(y)$.

The proof of the theorem of factorization given by Goursat, Cours d'analyse, 2, § 356, is based on Cauchy's analysis.

† Poincaré, Paris, Thèse, 1879, pp. 6, 7. Neumann, *Leipziger Berichte*, 35 (1883), p. 85; Abelsche Integrale, 2d ed., 1884, p. 125.

$(2n-1)$-dimensional manifolds, largely arbitrary in location and character, but necessarily passing through the loci of branch points, i. e., the branch manifolds, and along these junctions one branch of the function goes over into another branch, remaining analytic all the while.

A simple example or two will serve to illumine the above relations.

Example 1.—

$$u^2 - x = 0,$$

the independent variables being two in number, x and y. Here, the space of the independent variables is a four-dimensional real space R_4, corresponding to the two spheres,—the x-sphere and the y-sphere. If we set $x = x_1 + ix_2$, $y = y_1 + iy_2$, the points of R_4 will be (x_1, x_2, y_1, y_2). The Riemann manifold Φ is two-leaved. The branch manifold consists of the surfaces:

$$S_2': \quad (0, 0, y_1, y_2); \qquad S_2'': \quad (\infty, \infty, y_1, y_2),$$

where the point (y_1, y_2) ranges over the whole y-sphere.

As the junction we may take the 3-dimensional manifold

$$R_3: \qquad (x_1, 0, y_1, y_2), \quad 0 \leqq x_1 \leqq \infty,$$

i. e., the point (x_1, x_2) is any point of the positive axis of reals, including the points $x = 0$ and $x = \infty$; and (y_1, y_2) ranges independently over the extended y-plane. There is wide latitude in the choice of R_3, but it must contain the surfaces S_2' and S_2''.

Example 2.—

$$u^2 - xy^2 = 0,$$

the independent variables again being x and y.

Here, the two values of u become equal, not only in the points of the above surfaces S_2' and S_2'', but also in the points

$$\mathfrak{S}_2': \quad (x_1, x_2, 0, 0); \qquad \mathfrak{S}_2'': \quad (x_1, x_2, \infty, \infty).$$

Nevertheless, in the neighborhood of any point of $\mathfrak{S}_2'$ and $\mathfrak{S}_2''$ which does not lie on S_2' or S_2'' the values of u can be grouped so as to yield two functions, each single-valued and analytic throughout the neighborhood in question.

The Riemann manifold Φ may be taken precisely as before.

Let

$$\Delta(x_1, \cdots, x_n)$$

be the discriminant of F. Then $\Delta \not\equiv 0$, and to every point (x) in the neighborhood of $(x) = (0)$ in which $\Delta \neq 0$ there correspond m distinct roots of F. These may be so grouped as to yield m functions $u_1, \cdots, u_m$, each analytic in a preassigned point in which $\Delta \neq 0$. Moreover, one of these functions can be continued analytically into every other one, and thus they are all elements of one and the same monogenic analytic function.

If $n = 1$, we are led to an ordinary Riemann's surface with a single branch point in the point $x = 0$, in which all m leaves hang together.

If $n > 1$, it is still convenient to think of a Riemannian manifold Φ of m sheets, or leaves, as we will still say; though these leaves are no longer surfaces, but $2n$-dimensional manifolds.

We meet here, however, an entirely new order of relations. In the case $n = 1$, there was but a single branch point. That was fixed, and the junction lines were movable and to a large extent arbitrary. Here, however, the whole locus

$$(2) \qquad \Delta(x_1, \cdots, x_n) = 0$$

yields points for which two or more of the u_k's coincide. In such a point, two u's which coincide may or may not belong to functions each analytic at the point in question and satisfying the equation $F = 0$. In the former case, the Riemann manifold Φ has a multiple $(2n - 2)$-dimensional manifold, like a multiple point of a plane curve at which all the tangents are distinct and non-vertical, or more generally, at which no two branches are connected with each other.

In the latter case, however, we have a whole $(2n - 2)$-dimensional manifold of branch points, and the corresponding u's are not analytic in (x) at such a point. In other leaves above or below such a point it may, of course, happen that the corresponding determinations of u are analytic.

There still remain, in addition, the junctions. These are

This case can always be attained by a linear transformation:

(4) $$x_n = X_n + \alpha u,$$

where α is a suitable real positive number.

In fact, returning to the arbitrary case of the text, let P_1: $(x_1{}^1, \cdots, x_{n-1}{}^1)$ be a point of the neighborhood of P_0 in which the discriminant of no G_k vanishes; let this be true not merely for the particular element x_n' that was substituted in F, but for each of the other $l - 1$ determinations of x_n given by (3), $x_n'', \cdots, x_n{}^{(l)}$. Finally, let P_1 be so chosen that no two G's,—whether they belong to the same $x_n{}^{(i)}$ or to different ones: $x_n{}^{(i)}$ and $x_n{}^{(j)}$, have equal roots.

Throughout a certain neighborhood of P_1, then, we have l patches $\Sigma_1, \cdots, \Sigma_l$ of the discriminant manifold (3), and the points $(x_1, \cdots, x_{n-1}, x_n{}^{(k)})$, $k = 1, \cdots, l$, thus defined in the space of $(x_1, \cdots, x_n)$ are the totality of the points of so much of that space as lies in a certain neighborhood of the origin:

$$|x_j| < h, \qquad j = 1, \cdots, n,$$

for which, first, $(x_1, \cdots, x_{n-1})$ lies in the neighborhood of P_1 and, second, (1) has multiple roots. Let us picture the corresponding points $(u, x_1, \cdots, x_n)$ in a $(2n + 2)$-dimensional space, where u is given by the vanishing of the different $G_k(u, x_1, \cdots, x_{n-1})$, these functions being taken not merely for x_n', but also for the other $x_n{}^{(k)}$'s. Then these $(2n - 2)$-dimensional loci,—call them $\mathfrak{S}_1, \cdots, \mathfrak{S}_t$,—are distinct from one another. They are analogous to arcs of curves lying on the different nappes of cylinders whose directrices are the l $2n$-dimensional manifolds defined by (3):

$$(x_1, \cdots, x_{n-1}, x_n{}^{(k)}), \qquad k = 1, \cdots, l,$$

where $(x_1, \cdots, x_{n-1})$ lies in the neighborhood of P_1.

If now we make the transformation (4), restricting suitably the positive number α, then the configuration $F = 0$ will go over into a new configuration $F_1 = 0$ and at the same time the loci $\mathfrak{S}_1, \cdots, \mathfrak{S}_t$ will go over into loci $\mathfrak{S}_1', \cdots, \mathfrak{S}_t'$ which have for

§ 4. Continuation. The Branch Points of the Discriminant

It is important to notice how the dependent variable behaves in the points of a $(2n-2)$-dimensional manifold of branch points. If we are at liberty to make, if necessary, a non-singular linear transformation of the x's, we may assume that

$$\Delta(0, \cdots, 0, x_n) \not\equiv 0,$$

and hence replace the equation $\Delta = 0$ by an algebroid equation in x_n. Let

(3) $$D = x_n{}^l + B_1 x_n{}^{l-1} + \cdots + B_l = 0,$$

where D is an irreducible factor of Δ; and let $D_1(x_1, \cdots, x_{n-1})$ be the discriminant of D. Then $D_1 \not\equiv 0$. For simplicity in the presentation, we confine ourselves to the case that Δ has no further irreducible factor.

Consider a point P_0: $(x_1{}^0, \cdots x_{n-1}{}^0)$ in which $D_1 \neq 0$. In the neighborhood of this point the roots of (3) can be grouped to l functions $x_n{}', x_n{}'', \cdots, x_n{}^{(l)}$ each analytic in the above point and all elements of the same monogenic analytic function.

If we substitute one of these elements, $x_n{}'$, in the coefficients of (1), the new polynomial,

$$\bar{F} = u^m + \bar{A}_1 u^{m-1} + \cdots + \bar{A}_m = 0$$

—where $\bar{A}_k(x_1, \cdots, x_{n-1})$ is analytic in the point $(x_1{}^0, \cdots, x_{n-1}{}^0)$ but does not necessarily vanish there,—will have a common factor with its allied polynomial

$$\bar{F}' = \frac{\partial \bar{F}}{\partial u}.$$

Consider the greatest common divisor of $\bar{F}$ and $\bar{F}'$. Let its irreducible factors be

$$G_k(u, x_1, \cdots, x_{n-1}), \qquad k = 1, \cdots, \nu.$$

In general, $\nu = 1$ and G_1 is linear in u.

are x, y, z. Let

$$U = \frac{u}{x}.$$

Then

$$U^2 - (y^2 - z^2)(y^2 - k^2z^2) = 0.$$

But in the points of the singular manifold $x = 0$, i. e., in the points $(u, x, y, z) = (0, 0, y, z)$ the limiting values of U do not form a function single-valued on that manifold.

Connectivity and the Riemann Manifold. In the preceding sections we have taken the Riemann manifold Φ for granted. But how do we know that it exists? Even for functions of a single complex variable this question, in the general case, was not simple. It is one of the fundamental problems of the theory to show that, to any monogenic analytic function of several complex variables, corresponds a Riemann manifold. One method of attack would be to prove the theorem for a properly restricted algebroid configuration, and then proceed as in the case of functions of a single variable.*

Consider so much of the algebroid configuration (1), § 3, as lies in the region $|u| < k$, $|x_i| < h_i$. The linear connectivity of the corresponding Riemann manifold is not necessarily unity, no matter how far k, h_i be restricted. For, in particular, F may be a non-specialized homogeneous polynomial, so that (1) is the equation of a non-specialized algebraic cone of degree m.

Parametric Representation im Kleinen. One other theorem we will mention,—the theorem of the parametric representation of an analytic configuration im Kleinen.

Let $G(z_1, \cdots, z_n)$ be a function of $z_1, \cdots, z_n$ analytic at the origin and vanishing there, but not vanishing identically. Then there exists a finite number of systems of equations

$$(1) \qquad z_k = \varphi_k(t_1, \cdots, t_{n-1}), \qquad k = 1, \cdots, n,$$

where $\varphi_k(t_1, \cdots, t_{n-1})$ is analytic at the origin $(t) = (0)$ and van-

* Cf. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d. ed., 1912, p. 747.

the transformed configuration $F_1 = 0$ the same meaning that $\mathfrak{S}_1, \cdots, \mathfrak{S}_\mathfrak{k}$ have for $F = 0$, i. e., $\mathfrak{S}_1, \cdots, \mathfrak{S}_\mathfrak{k}$ are invariant of the transformation. And now, if the neighborhood of P_1 is suitably restricted, the number of the ordinates u corresponding, for a given $(x_1, \cdots, x_{n-1})$, to the ith region $\mathfrak{S}_i'$, $i = 1, \cdots, \mathfrak{k}$, will reduce to unity. Thus the new l will equal $\mathfrak{k}$, and to each of the new points $(x_1, \cdots, x_{n-1}, X_n^{(i)})$ will correspond but a single u.

It thus appears, that, in general, on the $(2n - 2)$-dimensional analytic manifold or manifolds defined by the equation $\Delta = 0$ the multiple roots of F are single-valued and analytic except along certain $(2n - 4)$-dimensional analytic manifolds.

We can now proceed to these latter manifolds and prove a similar theorem for them; and so on.

The reasoning here used is akin to that employed in the proof of Weierstrass's theorem, § 8.

§ 5. Single-Valued Functions on an Algebroid Configuration

Let U be uniquely defined in the ordinary points of the algebroid configuration (1), i. e., the points in which u is not a multiple root of (1), and let it be analytic in such points. If, furthermore, U remains finite, then, in the above points,

$$U = \frac{G(u, x_1, \cdots, x_n)}{F'(u, x_1, \cdots, x_n)},$$

where G is analytic in the point $(0, 0, \cdots, 0)$ and vanishes when F' vanishes on the manifold. Moreover, U is an algebroid function of $(x_1, \cdots, x_n)$ in the neighborhood of the origin.

It would be a mistake, however, to think that when U satisfies the above hypothesis, the limiting values of U in the singular manifold behave as did the coordinates, u, x_n, etc., in the cases discussed in § 4. The following example will show what may arise. Let

$$F \equiv u^2 - x^2(y^2 - z^2)(y^2 - k^2z^2) = 0,$$

where k is real and $0 < k < 1$, and the independent variables

$$(3) \quad \begin{cases} F_1(w_1, \cdots, w_m, w_{m+1}) = w_{m+1}^N + A_1 w_{m+1}^{N-1} + \cdots + A_N = 0, \\ w_{m+j} = \dfrac{F_j(w_1, \cdots, w_m, w_{m+1})}{F_1'(w_1, \cdots, w_m, w_{m+1})}, \quad j = 2, \cdots, n-m, \end{cases}$$

where A_k is analytic in $w_1, \cdots, w_m$ at the origin and vanishes there, and F_j is analytic in $w_1, \cdots, w_m, w_{m+1}$ at the origin and vanishes there. $F_1(w_1, \cdots, w_m, w_{m+1})$ is irreducible at the origin, and

$$F_1' = \frac{\partial F_1}{\partial w_{m+1}}.$$

To each root $(z) \neq (0)$ of (1) lying in the neighborhood in question corresponds at least one system (3) such that F_1' does not vanish in the point (w) which is the image of (z). Conversely, each system of values (w) lying in a certain neighborhood of the origin and satisfying (3) yields a root (z) of (1) lying in the neighborhood of $(z) = (0)$.

When the conditions of the problem are such that all n variables $z_1, \cdots, z_n$ are coordinate, so that the transformation (2) is available, this theorem yields complete and satisfactory information regarding the solution of equations (1) im Kleinen.

The proof of the theorem is direct, and is given by means of the factor theorem and the algorithm of the greatest common divisor.

§ 7. Continuation. A General Theorem

It may happen that the variables with respect to which it is desired to solve may not be interchanged with the remaining variables, so that the factor theorem is not available. In this case the following theorem may be useful.* The proof is closely allied to that of Weierstrass's theorem, § 6. The case $n = 2$ is covered by a theorem of Bliss's.†

* The theorem is suggested by a theorem of Poincaré's, Thèse, 1879, Lemma IV, p. 14. It is not clear what Poincaré meant by the words: " . . . si les équations $\varphi_1 = \varphi_2 = \cdots = \varphi_p = 0$ restent distinctes quand on annule tous les x. . . ."

† Princeton Colloquium, 1909, published, 1913, p. 71.

ishes there, such that, throughout a certain neighborhood of that point, the numbers $z_1, \cdots, z_n$ thus defined are roots of G. And conversely, to each root of G within a certain neighborhood of the origin there corresponds at least one system (1) which yields this point; and in any such system there corresponds but one point (t) to the given (z), provided $(z) \neq (0)$.

For the case $n = 2$ the proof of this theorem is readily given both by the methods of Riemann and by those of Noether (developed originally for algebraic curves). For $n = 3$, after an unsuccessful attempt by Kobb, a proof was given by Black.* The latter's methods appear to suffice for the general case; but detailed algebraic work remains to be done. Weierstrass states the theorem as true in all cases, $n = n$.†

§ 6. Solution of a System of Analytic Equations. Weierstrass's Theorem

There is a second theorem of Weierstrass's‡ which is less well known than the factor theorem, but which deserves a prominent place in the theory. It is as follows.

Let

$$(1) \qquad G_1(z_1, \cdots, z_n), \quad \cdots, \quad G_l(z_1, \cdots, z_n), \qquad l < n,$$

be a system of functions each analytic at the origin and vanishing there, but not vanishing identically. Then the roots of these functions, regarded as simultaneous, which lie in the neighborhood of the origin can be represented as follows. A suitable non-singular linear transformation being made:

$$(2) \qquad z_k = a_{k1} w_1 + \cdots + a_{kn} w_n, \qquad k = 1, \cdots, n,$$

the values of w which correspond to roots of the original functions (1) will be given by a finite number of systems of equations of the following type:

* *Proceedings Amer. Acad. of Arts and Sci.*, 37 (1902), p. 281.

† Werke, 3, pp. 103–4.

‡ Werke, 3, pp. 79–80.

CORRECTION

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p. 195, lines 3 and 4, omit the words "of D and likewise distinct points." and replace the second paragraph by the following:

"These points are determined as follows: One or more algebroid configurations are given by equations of the type

$$X^m + B_1 X^{m-1} + \ldots + B_m = 0, \qquad \Sigma m = M, \tag{5}$$

where $B_k(x_1, \ldots, x_{p-1})$ is analytic at the origin and vanishes there. If $(x_1, \ldots, x_{p-1})$ does not lie on E, the roots of (5) are all distinct and analytic in $(x_1, \ldots, x_{p-1})$, and the further coordinates $u_1, \ldots, u_l, x_p$ which enter in $(u; x)$ to form the roots of (1) are single-valued and analytic on the configuration (5)."

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except, at most, for those points for which $(x_1, \cdots, x_{p-1})$ lies on E.

The points of E are those whose co-ordinates satisfy at least one of a finite number of equations

$$E_1(x_1, \cdots, x_{p-1}) = 0, \quad \cdots, \quad E_k(x_1, \cdots, x_{p-1}) = 0,$$

where E_k is analytic in the point $(x_1, \cdots, x_{p-1}) = (0, \cdots, 0)$ and vanishes there, and is irreducible.

We can now proceed to treat the points of E in a similar way, and so on.

The foregoing formulation is deficient in one respect. In excepting, as the first step, all points whose (x) belongs to D some points were lost which have not later been regained. Consider the multiply sheeted Riemann manifold corresponding to (4). For a given point of D one point at least of this manifold is to be excluded. But it may happen that points above or below this one, in other sheets, are such that u_l and the other u_k's will be analytic there. The number of such systems, $(u_1, \cdots, u_l; x_1, \cdots, x_p)$ will, however, be less than N.

It would be possible to give to this theorem a formulation more closely resembling that of Weierstrass's theorem, § 6.

Theorem. Let the functions

$$(1) \qquad \Phi_k(u_1, \cdots, u_l; x_1, \cdots, x_p), \qquad k = 1, \cdots, l,$$

be analytic in the point $(u; x) = (0; 0)$ and vanish there. Furthermore, let the l equations

$$(2) \qquad \Phi_k(u_1, \cdots, u_l; 0, \cdots, 0) = 0, \qquad k = 1, \cdots, l,$$

admit no other solution than $(u) = (0)$ in the neighborhood of this latter point. Then, to each point (x) in a certain neighborhood of $(x) = (0)$, with the exception of those which lie on a locus D presently to be considered, there will correspond N distinct points, $(u_1^{(j)}, \cdots, u_l^{(j)})$, $j = 1, \cdots, N$, and hence N distinct points $(u_1^{(j)}, \cdots, u_l^{(j)}; x_1, \cdots, x_p)$, such that the equations

$$(3) \qquad \Phi_k(u_1, \cdots, u_l; x_1, \cdots, x_p) = 0$$

are satisfied in these points. N is a fixed positive integer.

Moreover, these are the only points of the neighborhood of $(u; x) = (0; 0)$ in which these equations are satisfied and for which (x) does not lie on D.

These points are determined as follows. u_l is given by an algebroid equation having no multiple factors,

$$(4) \qquad u_l^N + A_1 u_l^{N-1} + \cdots + A_N = 0,$$

where $A_k(x_1, \cdots, x_p)$ is analytic in the point $(x) = (0)$ and vanishes there. If (x) does not lie on D, the roots of (4) are all distinct, and analytic in (x), and the further functions $u_1, \cdots, u_{l-1}$ which enter to form the roots of (1) are also single-valued and analytic on the analytic configuration, or configurations, (4) except at most for points for which (x) lies on D.

The points of D are those whose co-ordinates satisfy at least one of a finite number of equations

$$D_1(x_1, \cdots, x_p) = 0, \quad \cdots, \quad D_q(x_1, \cdots, x_p) = 0,$$

where D_k is analytic in the point $(x) = (0)$ and vanishes there, and is irreducible.

$$J_{k-1} \equiv \frac{\partial(J_{k-2}, \Phi_2, \cdots, \Phi_l)}{\partial(u_1, u_2, \cdots, u_l)} = 0, \qquad (u; x) = (0; 0);$$

$$J_k \equiv \frac{\partial(J_{k-1}, \Phi_2, \cdots, \Phi_l)}{\partial(u_1, u_2, \cdots, u_l)} \neq 0, \qquad (u; x) = (0; 0).$$

Then the hypotheses of the above theorem are fulfilled, and $N = k$.

§ 8. The Inverse of an Analytic Transformation

Let

$$\text{(1)} \qquad \begin{cases} x_1 = f_1(u_1, \cdots, u_n), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n = f_n(u_1, \cdots, u_n), \end{cases}$$

where $f_k(u_1, \cdots, u_n)$, $k = 1, \cdots, n$, is analytic in the point $(u) = (0)$ and vanishes there. If the Jacobian J of the f's does not vanish, it is well known that the equations can be solved uniquely for the u's in terms of the x's, the resulting functions being analytic at the point $(x) = (0)$.

To pass to the other extreme, if the Jacobian vanishes identically, there is a relation between the f's. More precisely, let

$$T: \qquad |u_k| < h, \qquad\qquad k = 1, \cdots, n,$$

be an arbitrary neighborhood of the point $(u) = (0)$. Then there is a point (a) of this neighborhood and a function $F(x_1, \cdots, x_n)$ which is analytic in (x) at the point $(x) = (b)$, $b_k = f_k(a_1, \cdots, a_n)$, $k = 1, \cdots, n$, and which has the following property:

$$\text{(2)} \qquad F(f_1, \cdots, f_n) \equiv 0,$$

where (u) is any point of the neighborhood of (a).

Thus the n functions $f_k(u_1, \cdots, u_n)$ are connected by an identical relation* (2).

The intermediate case, that J vanishes at the point $(u) = (0)$, but does not vanish identically, has been an object of investigation in recent years.

* Peano-Genocchi, Calcolo differenziale e integrale, p. 162. Bliss, Princeton Colloquium, p. 67, where it is shown that, when $n = 2$, the point (a) may be taken at (0).

Poincaré has given a further theorem,* which he regards merely as another form of the theorem of his Lemma IV, cited above. Under the hypotheses of the last-named theorem he states that the system of equations (3) can be replaced by an equivalent system:

$$\Psi_k(u_1, \cdots, u_l; x_1, \cdots, x_p) = 0, \qquad k = 1, \cdots, l,$$

in which Ψ_k, in addition to satisfying the conditions imposed on Φ_k, is a polynomial in $u_1, \cdots, u_l$.

Special Cases of the Foregoing Theorem. A special case of this theorem has recently been investigated by MacMillan.† It is evident, in the light of Weierstrass's theorem, that no one of the functions $\Phi_i(u_1, \cdots, u_l; 0, \cdots, 0)$ can vanish identically. Let $\Phi_i(u_1, \cdots, u_l; 0, \cdots, 0)$ be developed into a series of homogeneous polynomials of ascending degrees, and let the polynomial of lowest degree,—the *characteristic polynomial,* as MacMillan calls it,—be denoted by $\varphi^{(k_i)}(u_1, \cdots, u_l)$, its degree being k_i. MacMillan considers the case that the resultant R of the characteristic polynomials does not vanish.‡

Bliss§ has also given a treatment of this case and has obtained the result that, when $R \neq 0$, the number N has the value:

$$N = \prod_{i=1}^{l} k_i.$$

Another special case of the main theorem, has been investigated by Clements.‖ Let the Jacobian J vanish in the point $(u; x) = (0; 0)$, and furthermore let $J_1 = J$,

$$J_2 \equiv \frac{\partial(J_1, \Phi_2, \cdots, \Phi_l)}{\partial(u_1, u_2, \cdots, u_l)} = 0, \qquad (u; x) = (0; 0);$$

. . . .

* Mécanique céleste, vol. 1, p. 72.

† *Math. Ann.*, 72 (1912), p. 157.

‡ Cf. Bliss's critique of Poincaré's theorem and the results obtained by MacMillan, *Transactions Amer. Math. Soc.*, 13 (1912), p. 135.

§ L. c.

‖ *Bulletin Amer. Math. Soc.* (2), 18 (1912), p. 451; *Transactions Amer. Math. Soc.*, 14 (1913), p. 325.

LECTURE V

THE PRIME FUNCTION ON AN ALGEBRAIC CONFIGURATION

§ 1. The Algebraic Functions of Deficiency 1 and the Doubly Periodic Functions. Generalizations

1. *The Riemann's Surface as Fundamental Domain.* The algebraic functions of deficiency unity and their integrals are closely allied to the doubly periodic functions and their related functions, the theta and the sigma functions. It is one of the leading ideas which Riemann introduced into the theory and which has been further developed by Klein and his school that these two classes of functions, from a higher point of view, may all be considered as functions on one and the same foundation (Substrat), the Riemann's surface, idealized as a fundamental domain. Thus the n-leaved surface of deficiency 1 (or, more particularly, the two-leaved surface with four branch points) and the parallelogram of periods are, when regarded as fundamental domains on which functions with familiar properties may be defined, equivalent.

The Theta Function. The single function in terms of which the group of functions considered in this theory can be expressed is, when we make the parallelogram of periods and its congruent repetitions the domain of the independent variable, the theta or the sigma function. The characteristic properties of this function are:

(a) that it is single-valued and analytic within and on the boundary of the parallelogram;

(b) that its values in corresponding points of the boundary are related to each other in a simple manner, namely,

$$\sigma(u+\omega) = -\, e^{\eta\left(u+\frac{\omega}{2}\right)}\sigma(u),$$

$$\sigma(u+\omega') = -\, e^{\eta'\left(u+\frac{\omega'}{2}\right)}\sigma(u);$$

First, let us observe, a general solution of the problem is given by the theorem of § 7 for all transformations (1) which are such that the point $(u) = (0)$ is the only point in this neighborhood which corresponds to $(x) = (0)$.

For the case $n = 2$ Clements discussed completely the above transformation under this last-named hypothesis. Moreover, his theorem cited in § 7, and Bliss's results apply to certain classes of transformations of the kind under consideration. Urner* and Dederick† have also studied the problem from a different point of view,—that of the effect of the transformation on certain curves which abut on a point where the Jacobian vanishes. Dederick‡ introduced the determinant J_2 (§ 7) in the case $l = 2$, and Urner extended the definition to the higher J's.

* *Transactions Amer. Math. Soc.*, 13 (1912), p. 232.

† Ibid., 14 (1913), p. 143.

‡ Harvard Thesis, 1909.

present themselves without difficulty. But they do not yield a transition to a new fundamental domain on which a function with the essential properties of the theta function is readily defined. Riemann constructed functions in a measure akin to the elliptic thetas by means of the theta functions of p arguments. But, aside from the fact that his functions have in general p roots,—never a single simple root,—they may in particular vanish identically.

Weierstrass, on the other hand, introduced functions which are single-valued and in general analytic on F, but which have a finite number of essential singularities.

A way out was found by Klein in the use of homogeneous variables, already employed by Aronhold and Clebsch in the study of transcendental functions.* Klein perceived still greater possibilities in these ideas and carried through the definition of a function which, considered on the allied manifold in the space of the homogeneous variables,—allied, I mean, to the Riemann's surface, F,—is a generalization of the elliptic theta function, namely, his *prime function*† $\Omega(x_1, x_2; y_1, y_2)$. The latter is a function, not of two variables x, y corresponding to two points of the given algebraic configuration, one of which, y, may be thought of as a parameter, x being the variable; but of *four* independent variables x_1, x_2, y_1, y_2. It is homogeneous in x_1, x_2, and also in y_1, y_2.

In Klein's investigations there are two very distinct things which he desires to accomplish. He wishes, it is true, to find a generalization of the elliptic theta function. But he also wishes to obtain a function which will formally be invariant of certain linear transformations,—often the collineations of the space in which the basal algebraic configuration (Grundkurve) is interpreted.

To accomplish the latter end, the value of the homogeneous

* This method was expounded systematically in Clebsch and Gordon's Abelsche Funktionen, of the year 1866.

† Göttingen lectures on the Abelian functions, S.-S., 1888, to S.-S., 1889; *Math. Ann.*, 36 (1889), p. 1.

(c) that it has a single root of the first order in the parallelogram.

These properties can be followed with ease when the function is transplanted to the two-leaved (or n-leaved) algebraic surface F, spread out over the z-plane. If a system of cuts is made in F so that a simply connected surface F' with a boundary is generated, a branch of the theta function will be single-valued in F'. This branch will be analytic in all the ordinary points of the surface and continuous in the branch-points, and its values on opposite sides of a cut will differ from each other by a factor always finite and different from zero. The point ∞ plays no exceptional rôle. Finally, the branch in question will have a single zero of the first order in F'.

If the function is considered on F, it will be infinitely multiple-valued. But in the neighborhood of any point its values can be grouped to branches each single-valued there, analytic in the ordinary points, and continuous in a branch point.

The Independent Variable. I spoke above of the *single* function in terms of which the group of functions considered in this theory can be expressed. But a function implies an independent, as well as a dependent, variable, and the theta function in the ordinary, restricted, sense is simple because of a felicitous choice of the independent variable. If we follow this variable over the fundamental domain taken now as the n-leaved surface F', we find in it a function on this domain,

(a) which is everywhere analytic in the ordinary points and continuous in the branch-points and at infinity;

(b) which takes on boundary values differing by an additive constant across a cut; and

(c) which maps the neighborhood of any point,—even though this be a branch point,—on a smooth single-leaved patch in the other plane. —The function happens to be in this case the everywhere finite integral of the algebraic configuration.

Generalizations for $p > 1$. On an n-leaved algebraic surface of deficiency $p > 1$ the algebraic functions and their integrals

borhood of a given point of the configuration, to what would appear in Weierstrass's theory as $\chi(t)dt$, where t denotes the parameter by means of which the neighborhood in question is uniformized, and $\chi(t)$ is analytic and does not vanish there.

Let $P_{\xi\eta}(x)$ be an Abelian integral of the third kind with its logarithmic discontinuities in the points $x = \xi$, $x = \eta$, and let

$$P_{\xi\eta}^{xy} = P_{\xi\eta}(x) - P_{\xi\eta}(y).$$

Moreover, $P_{\xi\eta}(x)$ shall be so chosen that

$$P_{\xi\eta}^{xy} = P_{xy}^{\xi\eta}.$$

Klein defines his prime function $\Omega(x_1, x_2; y_1, y_2)$ as the following limit:

$$\Omega(x_1, x_2; y_1, y_2) = \lim_{dx=0,\ dy=0} \sqrt{d\omega_x\, d\omega_y\, e^{-P_{x,\,y}^{x+dx,\ y+dy}}}\,.$$

We can now state the definition of the prime function which we propose to develop in detail in the following paragraphs. Let the algebraic configuration be an arbitrary one of deficiency $p > 1$, and let it be uniformized by automorphic functions with limiting circle in the t-plane.* Let the integral P, transferred to the t-plane, be written

$$P_{t,\,\tau}^{t',\,\tau'}.$$

Then

$$\Omega(t, \tau) = \lim_{\Delta t=0,\ \Delta\tau=0} \sqrt{\Delta t \Delta\tau\, e^{-P_{t,\,\tau}^{t+\Delta t,\ \tau+\Delta\tau}}}\,. \tag{1}$$

In form, then, the definition is identical with Klein's.† But whereas Klein's $d\omega_x$ is single-valued on a homogeneous configuration corresponding to the given algebraic configuration, our dt is not invariant of the transformations of the automorphic group. Transferred to the surface F it is infinitely multiple-valued.

On the other hand, $\Omega(t, \tau)$ is a function of the two independent variables t, τ, each being chosen arbitrarily in the fundamental

* The details of this uniformization are set forth in the second edition of the author's Funktionentheorie, vol. 1, 1912, ch. 14.

† *Math. Ann.*, 36 (1889), p. 12. Cf. also Klein's account of the relation of his prime function to Weierstrass's $E(x, y)$ and Schottky's $E(\xi, \eta)$; ibid., p. 13.

variables he employs is unquestioned. But in so proceeding, the former object is made secondary,—at least, the homogeneous variables must be accepted from the outset, and he does not obtain in the end a function of a *single* variable and a *single* parameter, like the elliptic theta transplanted to the surface F.

I propose to give here a direct solution of the former problem.

What has all this to do with functions of several complex variables? Just this, that the methods of that theory yield proofs where proofs, in the theory as developed by Klein, are lacking.

One word as to the importance of this mode of treatment. The algebraic functions and their integrals occupy a central position in analysis through their relation to the geometry of algebraic curves and surfaces, the theory of linear differential equations of the second order with algebraic coefficients, and the automorphic functions of one and of several variables, including the periodic functions of several variables. The progress of mathematics in the future, even more than in the past, will depend on the rapidity with which the recruits can be despatched to the frontier. As a result of the theorems of uniformization now rigorously established an improved treatment of the algebraic functions and their integrals has become possible and, by reason of its simplicity and generality, appears suited to supercede the methods hitherto used. In this treatment, the prime function as developed in the following paragraphs is the dominating factor and may be made the basis for the whole theory.

§ 2. The Prime Function Defined as a Limit

Generalizing from the elliptic case considered by Aronhold and the hyperelliptic case, which he himself had treated at length, Klein introduced, for an arbitrary algebraic configuration, reduced by birational transformation to a normal form, an expression which he called an "everywhere finite differential," and which he writes as $d\omega_x$. It is sufficient for our present purposes to know that this expression is analogous, for the neigh-

where $z = x + yi$, $\xi = \xi' + \xi''i$, $\eta = \eta' = \eta' + \eta''i$. Then

$$v = \theta + f(x, y), \tag{2}$$

where $f(x, y)$ is harmonic throughout K.

By means of this function:

$$v = v(x, y; \xi', \xi''; \eta', \eta''),$$

$2p$ everywhere finite logarithmic potentials can be constructed, each admitting the modulus of periodicity 1 across one of the $2p$ cross cuts, but being single-valued across each of the others. In fact, let C be a loop cut not passing through a branch point. Mark on C n points ξ_k, $k = 1, \cdots, n$, so chosen that about two consecutive points a circle K can be drawn. The function v being formed for each pair of consecutive points, the sum of these n functions will be an everywhere finite logarithmic potential with modulus of periodicity 2π across C.

These $2p$ functions are seen to be linearly independent. Out of them a normal system of p everywhere finite integrals can now be constructed:

(3)

	A_1,	A_2	$\cdots$	A_p	B_1	$\cdots$	B_p
w_1	πi	0	$\cdots$	0	a_{11}	$\cdots$	a_{1p}
w_2	0	πi	$\cdots$	0	a_{21}	$\cdots$	a_{2p}
$\vdots$	.	.	.	.	.	.	.
w_p	0	0	$\cdots$	πi	a_{p1}	$\cdots$	a_{pp}

where $a_{kl} = a_{lk}$.

Furthermore, if we denote the conjugate of the above function v by $-u$, then

$$u + vi$$

is an integral of the third kind. Such an integral can be obtained for an arbitrary pair of ordinary points ξ, η of the surface by joining these points by a curve L, interpolating on L $n - 1$ points so that two consecutive points lie in a circle K, forming the foregoing function for each pair of consecutive points, and adding these n functions together.

domain corresponding to the given algebraic configuration, and not on an allied configuration in the space of the homogeneous variables. Here, τ plays the rôle of a parameter, t being the (single) independent variable.

It is, however, important to know the nature of the dependence of Ω on both arguments, regarded as independent variables. So far as the analytic character of the dependence is concerned, theorems in the newer theory of functions of several complex variables afford precisely the tools that are needed.

§ 3. The Existence Theorems

It is evident that, if we are to deal with such a limit as the one here considered and infer the analytic properties of the limiting function, it will not suffice to study the function $P_{\xi\eta}^{xy}$ or $P_{t\tau}^{t'\tau'}$ merely in its dependence on one variable at a time. The theorem of II, § 5, combined with the existence theorems as developed by Neumann, enables us to obtain with ease the foundation needed.

Let us consider first, as lying nearest to the theory of Neumann,* an arbitrary algebraic Riemann's surface F and two points ξ, η of the same. Moreover, ξ, η shall be ordinary points, and it shall be possible to enclose them within a circle K not including any branch-point in its interior or on its boundary.

Let ξ, η be joined by a right line L, and let F be cut along L. Then there exists a logarithmic potential function v, single-valued and finite in the severed surface, harmonic at all ordinary points and continuous in the branch-points and at infinity, admitting harmonic continuation across L from either side, and such that

$$v|_{L^+} = v|_{L^-} + 2\pi.$$

Let θ be defined in the region consisting of K cut open along L as follows:

$$\begin{aligned}\theta &= \operatorname{arc}(z-\xi) - \operatorname{arc}(z-\eta)\\ &= \tan^{-1}\frac{y-\xi''}{x-\xi'} - \tan^{-1}\frac{y-\eta''}{x-\eta'},\end{aligned}$$

$$-\pi < \theta < \pi,$$

* Abelsche Integrale, 2d ed., 1884, chs. 16–18.

where $\varphi_k(z)$ denotes the integrand of the normal integral $w_k(z)$:

$$w_k(z) = \int_{z_0}^{z} \varphi_k(z)\,dz + \text{const.}$$

§ 4. Dependence on the Parameter

The function v of § 3, (2) has hitherto been considered solely in its dependence on x and y. It has the following further property, as is seen directly from the existence proof. Let K' be a circle concentric with K and of smaller radius, and let the points ξ, η be restricted to the interior of K'. Then $f(x, y)$, which now becomes a function of ξ', ξ'', η', η'' as well as of x, y, remains finite when (x, y) ranges over K and (ξ', ξ''), (η', η'') range independently over K', provided that we complete its definition by demanding, for example, that it vanish in a fixed point of K exterior to K'. The function is defined only when the points (ξ', ξ''), (η', η'') are distinct.

It is now readily inferred from the well-known properties of the logarithmic potential that the function conjugate to $f(x, y)$ also remains finite when (x, y), (ξ', ξ''), (η', η'') vary as above, the definition of this function being completed, for example, by demanding that it vanish in the same fixed point of K.

Finally, it is seen that v remains finite in the part of F exterior to K when ξ, η range over K'. Hence a branch of $u + vi$, considered in a simply connected region S of F, remains finite there, provided ξ, η are exterior to S and, moreover, that their minimum distance from the boundary of S does not fall below a certain positive constant. Similarly, the moduli of periodicity of $u + vi$ across the A and the B cuts remain finite when ξ, η range over K'.

Let ξ_0, η_0 be two distinct points of K', and let z_0, w_0 be two ordinary points of F distinct from ξ_0, η_0. Consider the cylindrical region $T = (T_\xi, T_\eta, T_z, T_w)$ corresponding to small circles about each of the points ξ_0, η_0, z_0, w_0. If three of the four variables are assigned arbitrary values in their circles and then held fast,

Finally, this integral can be reduced to a normal integral $\Pi_{\xi\eta}(z)$ with vanishing moduli of periodicity across the A cuts:

$$\begin{array}{c|ccc|ccc} & A_1 & \cdots & A_p & B_1 & \cdots & B_p \\ \hline \Pi_{\xi\eta}(z) & 0 & \cdots & 0 & 2w_1^{\xi\eta} & \cdots & 2w_p^{\xi\eta} \end{array}, \tag{4}$$

where $w_k(z)$ denotes a branch of the function taken in the simply connected surface F', and

$$w_k^{\xi\eta} = w_k(\xi) - w_k(\eta). \tag{5}$$

The integral $\Pi_{\xi\eta}(z)$ is completely determined save as to an additive constant, which is any function of ξ, η.

If we set

$$\Pi_{\xi\eta}^{zw} = \Pi_{\xi\eta}(z) - \Pi_{\xi\eta}(w), \tag{6}$$

then*

$$\Pi_{\xi\eta}^{zw} = \Pi_{zw}^{\xi\eta}. \tag{7}$$

The scheme of the moduli of periodicity of the function Π_{η}^{xy}, when regarded as a function of one variable at a time, is as follows.†

$$\begin{array}{c|ccc|ccc} & A_1 & \cdots & A_p & B_1 & \cdots & B_p \\ \hline x & 0 & \cdots & 0 & 2w_1^{\xi\eta} & \cdots & 2w_p^{\xi\eta} \\ y & 0 & \cdots & 0 & -2w_1^{\xi\eta} & \cdots & -2w_p^{\xi\eta} \\ \xi & 0 & \cdots & 0 & 2w_1^{xy} & \cdots & 2w_p^{xy} \\ \eta & 0 & \cdots & 0 & -2w_1^{xy} & \cdots & -2w_p^{xy} \end{array} \tag{8}$$

In a similar manner, the normal integral of the second kind is obtained:‡

$$Z_\xi(z) = \frac{1}{z-\xi} + \mathfrak{A}(z), \tag{9}$$

where ξ is an ordinary point and $\mathfrak{A}(z)$ is analytic at $z = \xi$.

$$\begin{array}{c|ccc|ccc} & A_1 & \cdots & A_p & B_1 & \cdots & B_p \\ \hline Z_\xi(z) & 0 & \cdots & 0 & -2\varphi_1(\xi) & \cdots & -2\varphi_p(\xi) \end{array} \tag{10}$$

* Cf. Appell et Goursat, Fonctions algébriques, p. 327.

† In these formulas, x, y denote complex variables.

‡ Neumann, Abelsche Integrale, 2d ed., 1884, p. 206.

The fundamental domain $\mathfrak{F}$ for the automorphic group G in question can be chosen as a circular polygon of $4p$ sides joined in pairs. In the canonical system of cuts in the n-leaved surface the positive direction of a B-cut shall be oriented to the positive direction of an A-cut as the positive axis of ordinates to the positive axis of abscissas; and the left bank of each cut, when the latter is described in the positive sense, shall be taken as the positive bank. The four banks appear, then, in the t-figure as indicated. We shall regard the points of the curves A_α^-, B_α^- as pertaining to $\mathfrak{F}$; those of A_α^+, B_α^+ as not pertaining to $\mathfrak{F}$.

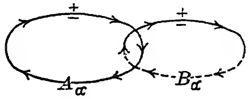

FIG. 4.

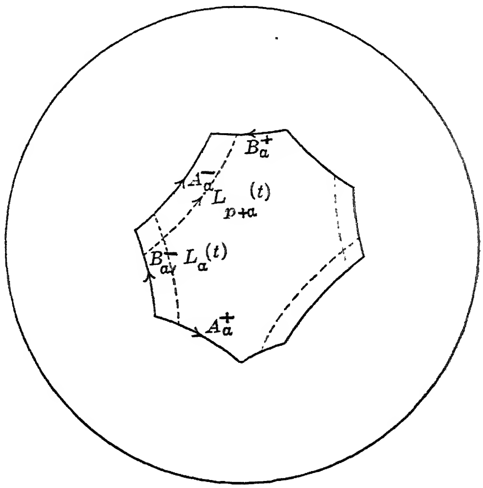

FIG. 5.

The Normal Integral of the Third Kind. In building up the integral of the third kind we have excluded the case that ξ, η lie at a branch point or at infinity. Such points present no peculiarity in the t-figure. We can remove this exception in either one of two ways: (a) we can go back and extend the earlier considerations to the cases excepted, or (b) we can establish the existence theorems directly for the case of the fundamental domain $\mathfrak{F}$.

There is no difficulty in carrying through the first method.

while the fourth is allowed to range over its circle, it appears from (6) and (7) that $\Pi_{\xi\eta}^{zw}$ is analytic in this variable alone. Furthermore, from the considerations which have just preceded, it is seen that when all four variables range over their circles, $\Pi_{\xi\eta}^{zw}$ remains finite in T. We infer, then, from the theorem of II, § 5, in its *restricted* form that Π is analytic in all four variables regarded as simultaneous.

Next, let us consider the function Π when $z_0 = \xi_0$, the point z lying in the circle about ξ_0. The points ξ_0, η_0, w_0 are distinct ordinary points. But it is necessary now to demand that z shall not coincide with ξ. If we write

$$\Pi_{\xi\eta}^{zw} = \log (z - \xi) + \mathfrak{A}(z, w, \xi, \eta), \tag{11}$$

then $\mathfrak{A}$ is defined at all points of T except those of the locus $z = \xi$, and $\mathfrak{A}$ is finite. It follows here, as in the earlier case, that $\mathfrak{A}$ is analytic in those points of T in which it is defined. And now comes a typical application of the theorem of III, § 4, relating to removable singularities. From it we infer that $\mathfrak{A}$ approaches a limit in each of the excepted points, and that, if $\mathfrak{A}$ is defined there as equal to its limit, then $\mathfrak{A}$ will be analytic there.

Similar formulas hold for other coincidences of the points z_0, w_0, ξ_0, η_0. Thus, when all four points coincide,

$$\Pi_{\xi\eta}^{zw} = \log \frac{(z - \xi)(w - \eta)}{(z - \eta)(w - \xi)} + A(z, w, \xi, \eta), \tag{12}$$

where A is analytic in all four arguments, regarded as simultaneous, in the point in question.

§ 5. The Functions in the Automorphic Fundamental Domain

We proceed now to transfer all the functions from the n-leaved Riemann's surface of the z-plane to a fundamental domain $\mathfrak{F}$ in the unit circle of the t-plane. The relation between z and t shall be expressed by the equation

$$z = \varphi(t), \quad \text{or} \quad t = \omega(z). \tag{13}$$

than poles within the circle. Those of the third kind have logarithmic singularities.

Consider, in particular, a normal integral of the first kind, $w_k^{t\tau}$, where t, τ correspond respectively to z, w. Then, if the transformation (15) be performed on t,

$$(17)\qquad \begin{cases} w_k^{t_\alpha\tau} = w_k^{t\tau}, & k \neq \alpha; \\ w_\alpha^{t_\alpha\tau} = w_\alpha^{t\tau} + \pi i. \end{cases}$$

Corresponding to a transformation (16) we have

$$(18)\qquad w_k^{t_{p+\alpha},\,\tau} = w_k^{t\tau} + a_{k\alpha}.$$

If, on the other hand, τ is transformed and t held fast, the sign of the additive term is reversed.

It appears, then, that the scheme (3) applies to the integrals $w_k^{t\tau}$, $k = 1, \cdots, p$, when regarded as functions of t alone, the transformations being those of (15), (16). When these integrals are regarded as functions of τ alone, each term in the scheme (3) is replaced by its negative.

The scheme (8) applies, with the requisite changes in the letters, to the transformed integral $\Pi_{t'\tau'}^{t\tau}$.

Homomorphic Functions. Any Abelian integral u, when subjected to a transformation of the group G, experiences a transformation of the type

$$(19)\qquad u' = u + A,$$

where A is a constant. These functions come under the general class of functions which undergo a transformation of the form

$$(20)\qquad \left(u \,\middle|\, \frac{Mu + N}{Pu + Q}\right), \qquad MQ - NP \neq 0,$$

where $M, \cdots, Q$ are constants. The general class of functions which have this property are called *homomorphic functions.**

To the algebraic functions on F correspond functions u of t, for which (20) reduces to the identity. These functions are

* Klein, Ueber lineare Differentialgleichungen zweiter Ordnung, Göttingen, S.-S., 1894 (lithographed), p. 492.

The second method,* however, yields the desired results with essentially the same machinery as that used for the n-leaved surface, but with no excepted cases whatever.

The final result in either case is that the formulas for which (11) and (12) are representative hold unchanged for the t-plane.

Thus, for example, if t, τ, t', τ' be any four distinct points of $\mathfrak{F}$, we shall have†

$$(14) \qquad \Pi_{t'\tau'}^{t\tau} = \log \frac{(t - t')(\tau - \tau')}{(t - \tau')(\tau - t')} + \mathfrak{A}(t, \tau; t', \tau'),$$

where $\mathfrak{A}(t, \tau; t', \tau')$ is analytic in its four arguments throughout the cylindrical region $(\mathfrak{F}, \mathfrak{F}, \mathfrak{F}, \mathfrak{F})$ inclusive of all boundary points.

The Transformations of the Functions. Let

$$(15) \qquad t_\alpha = L_\alpha(t), \qquad \alpha = 1, \cdots, p,$$

be the linear transformation of G which carries A_α^- into A_α^+; and let

$$(16) \qquad t_{p+\alpha} = L_{p+\alpha}(t), \qquad \alpha = 1, \cdots, p,$$

carry B_α^- into B_α^+. The inverse,

$$t' = L_\alpha^{-1}(t),$$

will carry A_α^+ into A_α^-; and similarly for (16).

The transformations (15), (16) together with their inverses constitute a system of generators for the automorphic group G.

Any single-valued function on F which has no other singularities than poles goes over into a single-valued function of t having no other singularities than poles within the unit circle $|t| < 1$ and (with the single exception of a constant) having the circumference of the circle as a natural boundary.

The Abelian integrals of the first and second kind also go over into single-valued functions of t having no other singularities

* Cf. Fricke-Klein, Automorphe Funktionen, vol. 2, chap. 1.

† The notation for the dependent variable here and in the following is the same whether the arguments are taken in the t-circle or on the z-surface. It will be clear from the context each time which is meant.

The Behavior of $X(t, \tau)$ *on the Boundary.* Let τ be an interior point of $\mathfrak{T}$, and let c be a point of B_a^-, c_a its image on B_a^+:

(21) $$c_a = L_{p+a}(c).$$

Then, by (8),

$$\begin{aligned} \Pi^{c_a\tau}_{t'\tau'} &= \Pi^{c\tau}_{t'\tau'} + 2w_a^{t'\tau'} \\ &= \Pi_{c\tau}(t') - \Pi_{c\tau}(\tau') + 2w_a^{t'\tau'}. \end{aligned}$$

Let $c' = c + \Delta c$ be a second point of B_a^- near c, $c'_a = c_a + \Delta c_a$ its image on B_a^+. We have, then, by the aid of the last equation, and (18), (8):

$$\begin{aligned} \Pi^{c_a\tau}_{c'_a\tau'} &= \Pi_{c\tau}(c'_a) - \Pi_{c\tau}(\tau') + 2w_a^{c'_a\tau'} \\ &= \Pi^{c\tau}_{c'\tau'} + 2w_a^{c\tau} + 2w_a^{c'\tau'} + 2a_{aa}. \end{aligned}$$

Turning now to the functions

$$X(c_a, \tau, \Delta c_a, \Delta\tau) = \Delta c_a \Delta\tau\, e^{-\Pi^{c_a\tau}_{c'_a\tau'}},$$

$$X(c, \tau, \Delta c, \Delta\tau) = \Delta c \Delta\tau\, e^{-\Pi^{c\tau}_{c'\tau'}}$$

it is seen that

$$X(c_a, \tau, \Delta c_a, \Delta\tau) = \frac{\Delta c_a}{\Delta c}\, e^{-2w_a^{c\tau} - 2w_a^{c'\tau'} - 2a_{aa}}\, X(c, \tau, \Delta c, \Delta\tau).$$

Let Δc, $\Delta\tau$ approach 0 as their limit. Then

(22) $$X(c_a, \tau) = \frac{dc_a}{dc} e^{-4w_a^{c\tau} - 2a_{aa}} X(c, \tau).$$

If c, c_a had been chosen on A_a^-, A_a^+, where now

$$c_a = L_a(c),$$

we should have had, corresponding to (22):

(23) $$X(c_a, \tau) = \frac{dc_a}{dc} X(c, \tau).$$

The Monogenic Analytic Function $X(t, \tau)$ *and its Transformations under the Group* G. We are now in a position to show that $X(t, \tau)$ can be continued analytically throughout the domain

absolute invariants of the group G, and are automorphic functions of t.

§ 6. An Auxiliary Function

Let t, τ be any two distinct points of the fundamental domain $\mathfrak{F}$. We can, without loss of generality, think of them as interior points, since the precise boundary of $\mathfrak{F}$ can at any point P be modified slightly by what is known as an "erlaubte Abänderung," the choice of the boundary as circular arcs being made merely for simplicity and definiteness.

Let $t' = t + \Delta t$ and $\tau' = \tau + \Delta\tau$ be two variable points of the neighborhoods respectively of t, τ. Form the function

$$X'(t, \tau, \Delta t, \Delta\tau) = \Delta t\,\Delta\tau\, e^{-\overline{\Pi}^{t\tau}_{t+\Delta t,\, \tau+\Delta\tau}}.$$

Then, by (14):

$$X'(t, \tau, \Delta t, \Delta\tau) = (t - \tau - \Delta\tau)(t - \tau + \Delta t)e^{\pi i - \mathfrak{A}(t,\, \tau;\; t+\Delta t,\, \tau+\Delta\tau)}.$$

But the function $\mathfrak{A}(t, \tau, t', \tau')$ is analytic in the point (t, τ, t, τ), and hence $X'(t, \tau, \Delta t, \Delta\tau)$ approaches a limit when Δt, $\Delta\tau$ independently approach 0 as their limit. We have, then,

$$\lim_{\Delta t=0,\, \Delta\tau=0} X'(t, \tau, \Delta t, \Delta\tau) = (t - \tau)^2 e^{\pi i - \mathfrak{A}(t,\, \tau,\, t,\, \tau)} = X(t, \tau).$$

The excepted points, $t = \tau$, are seen to be but removable singularities for the function X, and thus $X(t, \tau)$ is defined and is analytic throughout the cylindrical domain $(\mathfrak{F}, \mathfrak{F})$. If we regard τ as a parameter, X, considered as a function of t, vanishes twice in the point $t = \tau$ and nowhere else in $\mathfrak{F}$.

To obtain the function $X(t, \tau)$ throughout its entire domain of definition we could use the same limiting process as above, subjecting t and τ independently to transformations of G. It is sufficient, however, to know how X behaves on the boundary of $\mathfrak{F}$, or $(\mathfrak{F}, \mathfrak{F})$. In order to ascertain these relations, we will employ the method which is familiar when one is dealing with fundamental domains.*

* Osgood, Funktionentheorie, vol. 1, chap. 10, § 8.

(ix) $$\mathrm{X}(L_{p+a}^{-1}(t), \tau) = M'_{p+a}(t)\, e^{4w_a'^{t\tau} - 2a_{aa}}\mathrm{X}(t, \tau),$$

(x) $$\mathrm{X}(t, L_{p+a}^{-1}(\tau)) = M'_{p+a}(\tau)\, e^{-4w_a'^{t\tau} - 2a_{aa}}\mathrm{X}(t, \tau),$$

$$M'_{p+a}(t) = \frac{1}{L'_{p+a}[L_{p+a}^{-1}(t)]}.$$

It will be observed that $L'_a(t)$, $L'_{p+a}(t)$, $\alpha = 1, \cdots, p$, are analytic and different from zero throughout K.

§ 7. The Prime Function $\Omega(t, \tau)$

The prime function $\Omega(t, \tau)$ is now defined as follows.* The two square roots of the function $X(t, \tau)$ can be so grouped as to yield two functions each single-valued and analytic throughout (K). Let one of these functions be denoted by $\sqrt{\mathrm{X}(t, \tau)}$. Then

(A) $$\Omega(t, \tau) = C\sqrt{\mathrm{X}(t, \tau)},$$

where C denotes a constant not zero.

This function has the following characteristic properties, which are easily proven from its definition and the developments of the last paragraph.

(a) $\Omega(t, \tau)$ is analytic in the cylindrical region $(K) = (K, K)$ and has the boundary of this region as a natural boundary.

(b) $$\Omega(t, \tau) = -\,\Omega(\tau, t).$$

(c) $$\Omega(t, \tau) = (t - \tau)\, Q(t, \tau),$$

where $Q(t, \tau)$ is analytic in (K) and

$$Q(t, t) \neq 0.$$

More generally, let

$$t' = L(t)$$

* In the definition of $\mathrm{X}(t, \tau)$ it is not essential that the *normal* integral $\Pi_{t'\tau'}^{t\tau}$ be used. Any other integral of the third kind, $P_{t'\tau'}^{t\tau}$, which permits the interchange of parameters and arguments would serve equally well. Thus a prime function $\Omega_P(t, \tau)$ would result which has the same properties (a), (b), (c), but for which (d_1) is replaced by a pair of similar equations, each right hand side having an exponential factor whose exponent is a linear function of $w_1^{t\tau}, \cdots, w_p^{t\tau}$.

$(K) = (K, K)$, when K refers to the unit circle. Since the details are precisely parallel to those in the treatment of the σ-function, Funktionentheorie, Chap. 10, § 8, they may be left to the reader. We thus obtain a function analytic throughout this domain and not admitting analytic continuation beyond it. This function has the following properties.

(i) $$\mathrm{X}(t, \tau) = \mathrm{X}(\tau, t);$$

(ii) $$\mathrm{X}(t, \tau) = (t - \tau)^2 \Psi(t, \tau),$$

where $\Psi(t, \tau)$ is analytic throughout (K) and

$$\Psi(\tau, \tau) \neq 0;$$

(iii) $$\mathrm{X}(t_\alpha, \tau) = L'_\alpha(t) \mathrm{X}(t, \tau), \qquad \alpha = 1, \cdots, p;$$

(iv) $$\mathrm{X}(t_{p+\alpha}, \tau) = L'_{p+\alpha}(t)\, e^{-4w_\alpha^{t\tau} - 2a_{\alpha\alpha}} \mathrm{X}(t, \tau), \quad \alpha = 1, \cdots, p;$$

(v) $$\mathrm{X}(t, \tau_\alpha) = L'_\alpha(\tau) \mathrm{X}(t, \tau);$$

(vi) $$\mathrm{X}(t, L_{p+\alpha}(\tau)) = L'_{p+\alpha}(\tau)\, e^{4w_\alpha^{t\tau} - 2a_{\alpha\alpha}} \mathrm{X}(t, \tau).$$

If the transformation

$$t_\alpha = L_\alpha(t), \qquad \alpha = 1, \cdots, p,$$

is replaced by its inverse,

$$t' = L_\alpha^{-1}(t) = M_\alpha(t),$$

formulas (iii) and (v) become:

(vii) $$\mathrm{X}(L_\alpha^{-1}(t), \tau) = M'_\alpha(t) \mathrm{X}(t, \tau),$$

(viii) $$\mathrm{X}(t, L_\alpha^{-1}(\tau)) = M'_\alpha(\tau) \mathrm{X}(t, \tau),$$

$$M'_\alpha(t) = \frac{1}{L'_\alpha[L_\alpha^{-1}(t)]}.$$

But if

$$t_{p+\alpha} = L_{p+\alpha}(t), \qquad \alpha = 1, \cdots, p,$$

is replaced by its inverse,

$$t' = L_{p+\alpha}^{-1}(t) = M_{p+\alpha}(t),$$

formulas (iv) and (vi) are replaced by the following:

having the same properties. Let τ be an arbitrary point of $\mathfrak{F}$, and let it be held fast. Form the function

$$\frac{\Psi(t, \tau)}{\Omega(t, \tau)}.$$

Then this function, regarded as a function of t, will be analytic in K except for removable singularities in the points $t = \tau$ and in the images of this point under the group G. Let it be defined in these points as equal to its limit. The new function is analytic without exception in K, and does not vanish there.

From (d) it follows further that this function is invariant of the transformations of G. It is, therefore, a constant, as can be seen at once by transforming it to the n-leaved surface F. Hence

$$\frac{\Psi(t, \tau)}{\Omega(t, \tau)} = f(\tau), \qquad \Psi(t, \tau) = f(\tau)\, \Omega(t, \tau), \qquad f(\tau) \neq 0.$$

This last relation is an identity in t, τ, and hence can equally well be written in the form

$$\Psi(\tau, t) = f(t)\, \Omega(\tau, t).$$

Now apply the property represented by (b). It follows that

$$-\Psi(t, \tau) = -f(t)\, \Omega(t, \tau).$$

Hence

$$f(t) = f(\tau),$$

and this completes the proof.

From the foregoing result it is seen that the properties (a), $\cdots$, (d) can serve as the basis for an independent definition of the prime function $\Omega(t, \tau)$. Thus the function might be represented by an infinite product, as Weierstrass defined his elliptic σ-function. And just as Weierstrass made the latter the basal function for the whole theory of the elliptic functions, so the algebraic functions of deficiency $p > 1$, and their integrals, can be represented in terms of $\Omega(t, \tau)$. We proceed to give the fundamental formulas.

be any transformation of G. Then

$$\Omega(t, \tau) = (t - \tau')\, Q'(t, \tau),$$

where $Q'(t, \tau)$ is analytic in (K) and

$$Q'(t, t) \neq 0.$$

$$(d) \qquad \begin{cases} \Omega(t_a, \tau) = \sqrt{L_a'(t)}\, \Omega(t, \tau); \\ \Omega(t_{p+a}, \tau) = \sqrt{L_{p+a}'(t)}\, e^{-2w_a^{t\tau} - a_{aa}}\, \Omega(t, \tau). \end{cases}$$

Here, each square root denotes a function of t analytic in K and different from zero there. The value of these functions is given below.

Furthermore

$$(d') \qquad \begin{cases} \Omega(t, \tau_a) = \sqrt{L_a'(\tau)}\, \Omega(t, \tau), \\ \Omega(t, \tau_{p+a}) = \sqrt{L_{p+a}'(\tau)}\, e^{2w_a^{t\tau} - a_{aa}}\, \Omega(t, \tau), \end{cases}$$

where the square roots denote the same functions as above, in (d).

These square roots have the following determination. The transformations

$$t' = L_a(t), \quad t' = L_{p+a}(t)$$

are hyperbolic. Let them be written in the form

$$\frac{t' - \zeta_0}{t' - \zeta_1} = A\,\frac{t - \zeta_0}{t - \zeta_1}, \qquad 0 < A.$$

The fix points ζ_0, ζ_1 lie on the circumference of the fundamental circle. Then

$$\sqrt{L'(t)} = A^{\frac{1}{2}}\,\frac{L(t) - \zeta_1}{t - \zeta_1},$$

where $L(t)$ is any one of the functions $L_a(t)$, $L_{p+a}(t)$.

§ 8. The Determination of $\Omega(t, \tau)$ by Functional Equations

The properties (a), (b), (c), (d) or (d') serve to characterize the function Ω completely. For, let $\Psi(t, \tau)$ be a second function

variable, the second term on the right of the second formula in (A_1) is reversed in sign.

On the other hand, the formulas (A_2) are simplified.

$$(A_2') \qquad \begin{cases} \Pi^{st}_{\sigma_a \tau} = \Pi^{st}_{\sigma\tau}, \\ \Pi^{st}_{\sigma_{p+a}, \tau} = \Pi^{st}_{\sigma\tau} + 2w^{st}_a. \end{cases}$$

The new formulas (A_3') are similar, the last term being reversed in sign.

The Functions w^{st}_a, $\Phi_a(t)$. From (A_2') and II. we obtain:

$$\text{III.} \qquad w^{st}_a = \tfrac{1}{2}\log \frac{\Omega(s, c_{p+a})\,\Omega(t, c)}{\Omega(t, c_{p+a})\,\Omega(s, c)}$$

or

$$w^{ta}_a = \tfrac{1}{2}\log \frac{\Omega(t, c_{p+a})\,\Omega(a, c)}{\Omega(t, c)\,\Omega(a, c_{p+a})} = \tfrac{1}{2}\log \frac{\Omega(t, c_{p+a})}{\Omega(t, c)} + \text{const.}$$

The expression on the right-hand side is multiple-valued, but the different values can be grouped so as to yield single-valued functions each analytic in K. We choose that one of these functions which vanishes when $t = s$, or $t = a$.

A second expression for w^{st}_a as the integral of a single-valued function is given below, Formula VII.

Let $\Phi_a(t)$ be defined as follows:

$$\Phi_a(t) = \frac{dw^{ta}_a}{dt}. \qquad w^{st}_a = \int_t^s \Phi_a(t)\,dt.$$

Then

$$\text{IV.} \qquad \Phi_a(t) = \tfrac{1}{2}\frac{\partial}{\partial t}\log \frac{\Omega(t, c_{p+a})}{\Omega(t, c)}.$$

The Functions $Y_\tau(t)$, Y^{st}_τ. Let

$$\text{V.} \qquad Y_\tau(t) = \frac{\partial}{\partial \tau}\Pi_{\sigma\tau}(t) = -\frac{\partial}{\partial \tau}\log \Omega(t, \tau) = \frac{1}{t - \tau} + \mathfrak{A}(t, \tau).$$

If we differentiate (A_1) with respect to τ, we obtain

$$(B_1) \qquad \begin{cases} Y_\tau(t_a) = Y_\tau(t), \\ Y_\tau(t_{p+a}) = Y_\tau(t) - 2\Phi_a(\tau). \end{cases}$$

§ 9. The Abelian Integrals in Terms of the Prime Function*

The Functions $\Pi_{\sigma\tau}(t)$, $\Pi_{\sigma\tau}^{st}$. From (c) and (d) of § 7 it is seen that the formula

$$\text{I.} \qquad \Pi_{\sigma\tau}(t) = \log \frac{\Omega(t, \sigma)}{\Omega(t, \tau)}$$

gives a particular normal integral of the third kind, the general integral differing from the above by an additive term which is an arbitrary function of σ, τ. In the absence of any reason to the contrary we set this term equal to zero, i. e., we lay down arbitrarily the definition I. Thus this integral is completely determined.

If we set

$$t_\alpha = L_\alpha(t), \quad t_{p+\alpha} = L_{p+\alpha}(t), \qquad \alpha = 1, \cdots, p,$$
$$\frac{dt_\alpha}{dt} = t'_\alpha, \qquad \frac{dt_{p+\alpha}}{dt} = t'_{p+\alpha},$$

we find

$$A_1: \quad \begin{cases} \Pi_{\sigma\tau}(t_\alpha) = \Pi_{\sigma\tau}(t), \\ \Pi_{\sigma\tau}(t_{p+\alpha}) = \Pi_{\sigma\tau}(t) + 2w_\alpha^{\sigma\tau}; \end{cases}$$

$$A_2: \quad \begin{cases} \Pi_{\sigma_\alpha,\tau}(t) = \Pi_{\sigma\tau}(t) + \tfrac{1}{2}\log \sigma'_\alpha, \\ \Pi_{\sigma_{p+\alpha},\tau}(t) = \Pi_{\sigma\tau}(t) + \tfrac{1}{2}\log \sigma'_{p+\alpha} + 2w_\alpha^{t\sigma} - a_{\alpha\alpha}; \end{cases}$$

$$A_3: \quad \begin{cases} \Pi_{\sigma,\tau_\alpha}(t) = \Pi_{\sigma\tau}(t) - \tfrac{1}{2}\log \tau'_\alpha, \\ \Pi_{\sigma,\tau_{p+\alpha}}(t) = \Pi_{\sigma\tau}(t) - \tfrac{1}{2}\log \tau'_{p+\alpha} - 2w_\alpha^{t\tau} + a_{\alpha\alpha}. \end{cases}$$

The function $\Pi_{\sigma\tau}^{st}$ is now represented as follows:

$$\text{II.} \qquad \Pi_{\sigma\tau}^{st} = \log \frac{\Omega(s, \sigma)\,\Omega(t, \tau)}{\Omega(s, \tau)\,\Omega(t, \sigma)}.$$

For this function, regarded as a function of s alone, the formulas (A_1) hold without modification. When t is the sole independent

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* The deductions of this paragraph are suggested by corresponding formulas in the case $p = 1$, the function $\Omega(t, \tau)$ corresponding to $\vartheta(t - \tau)$ or $\sigma(t - \tau)$, Cf. Klein, *Math. Ann.* 36 (1889), p. 11.

The Functions $Y_\tau^{(m)}(t)$. From the relations

$$\frac{\partial Y_\tau(t)}{\partial \tau} = \frac{1}{(t-\tau)^2} + \mathfrak{A}_1(t, \tau),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial^{m-1} Y_\tau(t)}{\partial \tau^{m-1}} = \frac{(m-1)!}{(t-\tau)^m} + \mathfrak{A}_{m-1}(t, \tau),$$

where $\mathfrak{A}_{m-1}(t, \tau)$ is analytic within and on the boundary of $(\mathfrak{F}, \mathfrak{F})$, we are led to define $Y_\tau^{(m)}(t)$ as follows:

X. $$Y_\tau^{(m)}(t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} Y_\tau(t)}{\partial \tau^{m-1}}.$$

Here,

$$Y_\tau^{(m)}(t) = \frac{1}{(t-\tau)^m} + \mathfrak{B}(t, \tau).$$

From (B_1) we obtain the formulas:

$$(C) \qquad \begin{cases} Y_\tau^{(m)}(t_a) = Y_\tau^{(m)}(t), \\ Y_\tau^{(m)}(t_{p+a}) = Y_\tau^{(m)}(t) - \dfrac{2}{(m-1)!}\Phi_a^{(m-1)}(\tau). \end{cases}$$

§ 10. The Integral of the Second Kind on F

The function $Y_\tau(t)$, when transferred to the n-leaved surface F, is an integral of the second kind with a simple pole and with its moduli of periodicity corresponding to the A-cuts all zero. It differs, however, from the integral there taken as the normal integral, namely $Z_\xi(z)$, as follows.

We denoted by

$$z = \varphi(t)$$

the function which maps the n-leaved bounded surface F' on the fundamental domain $\mathfrak{F}$. Let τ be a point of $\mathfrak{F}$ corresponding to an ordinary point ξ of F'. Then

$$Z_\xi(z) = \frac{1}{z-\xi} + \mathfrak{A}(z), \qquad Y_\tau(t) = \frac{1}{t-\tau} + \mathfrak{B}(t),$$

where $\mathfrak{A}(z)$ and $\mathfrak{B}(t)$ are analytic respectively in the points $z = \xi$ and $t = \tau$.

If, furthermore, we differentiate (A_3) with respect to τ, we have

$$(B_2)\qquad \begin{cases} Y_{\tau_a}(t) = \dfrac{1}{\tau_a'}\, Y_\tau(t) - \dfrac{1}{2}\,\dfrac{\tau_a''}{\tau_a'^2}, \\[2ex] Y_{\tau_{p+a}}(t) = \dfrac{1}{\tau_{p+a}'}\, Y_\tau(t) + 2\,\dfrac{\Phi_a(\tau)}{\tau_{p+a}'} - \dfrac{1}{2}\,\dfrac{\tau_{p+a}''}{\tau_{p+a}'^2}, \end{cases}$$

where the accents denote differentiation.

Let Y_τ^{st} be defined as follows:

$$Y_\tau^{st} = Y_\tau(s) - Y_\tau(t).$$

For this function, regarded as a function of s, formulas (B_1) hold unchanged. When t is taken as the sole variable, the last term in these formulas is reversed in sign.

On the other hand, (B_2) is replaced by

$$(B_2')\qquad Y_{\tau_a}^{st} = \frac{1}{\tau_a'}\, Y_\tau^{st}, \qquad Y_{\tau_{p+a}}^{st} = \frac{1}{\tau_{p+a}'}\, Y_\tau^{st}.$$

From (B_1) it follows that

$$\text{VI.}\qquad \Phi_a(\tau) = \tfrac{1}{2}\{Y_\tau(t) - Y_\tau(t_{p+a})\}.$$

$$\text{VII.}\qquad w_a^{st} = \tfrac{1}{2}\int_t^s \{Y_\tau(c) - Y_\tau(c_{p+a})\}\,d\tau.$$

The Derivatives of $\Omega(t, \tau)$. Let

$$\Omega_1(t, \tau) = \frac{\partial\,\Omega(t, \tau)}{\partial t}, \qquad \Omega_2(t, \tau) = \frac{\partial\,\Omega(t, \tau)}{\partial \tau}.$$

From V. it follows that

$$\text{VIII.}\qquad \Omega_2(t, \tau) = -\,\Omega(t, \tau)\, Y_\tau(t).$$

From (b) we have:

$$\Omega_1(t, \tau) = -\,\Omega_2(\tau, t).$$

Hence

$$\text{IX.}\qquad \Omega_1(t, \tau) = -\,\Omega(t, \tau)\, Y_t(\tau).$$

Nor is this the only point at which this factor enters. We turn next to the functions which correspond to the adjoint C_{n-3}, namely, the integrands of the everywhere finite integrals, and we shall find the factor reappearing there, also. In § 14 we shall discuss at length the nature of this factor.

§ 11. The Integrands of the Integrals of the First Kind

If we write $w_a(z)$, considered on the surface F, in the form

$$w_a(z) = \int_{z_0}^{z} \varphi_a(z)\, dz,$$

and recall that, when w_a is considered in K,

$$w_a^{ta} = \int_{t_0}^{t} \Phi_a(t)\, dt + \text{const.},$$

we see that

$$\text{XII.} \quad \varphi_a(z) = \frac{1}{\varphi'(t)}\, \Phi_a(t), \qquad \text{or} \qquad \varphi_a(z) \Big/ \left(\frac{dt}{dz}\right) = \Phi_a(t).$$

The function $\varphi_a(z)$ is single-valued on F, and its only singularities are poles, which are situated in the branch points of F. $\Phi_a(t)$, on the other hand, is analytic without exception in $\mathfrak{F}$, or in K. Here again, then, it is the factor $1/(dt/dz) = \varphi'(t)$ that removes the singularities from an earlier function.

The functions $\Phi_a(t)$ are not invariants under the group G, nor are they even homomorphic. They take on a factor which is finite and different from zero. In fact,

$$\text{(D)} \qquad \Phi_k(t_a) = \frac{\Phi_k(t)}{L'_a(t)}, \qquad \Phi_k(t_{p+a}) = \frac{\Phi_k(t)}{L'_{p+a}(t)},$$

where $k = 1, \cdots, p$ and $\alpha = 1, \cdots, p$.

In the treatment of Abel's Theorem and the theorem of Riemann-Roch, when the functions entering were considered on F, there were exceptional cases that required special consideration, due to a function's becoming infinite. When the same theorems are treated on the fundamental domain $\mathfrak{F}$ and the method of contour integration is used, the results are completely general, no exceptions whatever occurring.

It follows, then, since

$$z - \xi = \varphi'(\tau)(t-\tau) + \frac{\varphi''(\tau)}{2!}(t-\tau)^2 + \cdots,$$

and $\varphi'(\tau) \neq 0$, that

$$\text{XI.} \qquad Z_\xi(z) = \frac{1}{\varphi'(\tau)} Y_\tau(t) + f(\tau).$$

This last result is fundamental in showing the unfortunate choice made in the earlier case in defining the normal integral of the second kind, whenever it is a question of the dependence of this integral on the parameter.* As ξ approaches a branch point of F, the integral $Z_\xi(z)$ is seen to approach no limit, and its moduli of periodicity across the B-cuts,—the integrands of the everywhere finite integral on F,—or certainly some of them, become infinite. Nevertheless $Z_\xi(z)$ has a perfectly definite value when ξ lies in a branch point, and the relation of the integral in this case to $Y_\tau(t)$ is obtained from the formulas

$$Z_\xi(z) = \frac{1}{(z-\xi)^{1/m}} + \mathfrak{A}(z),$$

$$z - \xi = \frac{\varphi^{(m)}(\tau)}{m!}(t-\tau)^m + \cdots, \qquad \varphi^{(m)}(\tau) \neq 0,$$

$$\text{XI}'. \qquad Z_\xi(z) = \left(\frac{m!}{\varphi^{(m)}(\tau)}\right)^{1/m} Y_\tau(t) + \text{const.}$$

There is, then, complete discontinuity in the dependence of $Z_\xi(z)$ on ξ at a branch point, and this discontinuity does not correspond to any important property of the integral of the second kind. On the other hand, the points τ for which $\varphi'(\tau) = 0$ are in no wise exceptional points for the function $Y_\tau(t)$.

It appears, then, that the earlier integral should have been normalized so as to correspond to $Y_\tau(t)$. This can be attained without reference to $Y_\tau(t)$ by replacing the earlier $Z_\xi(z)$ by

$$Z_\xi(z) \Big/ \left(\frac{dt}{dz}\right)_{z=\xi}.$$

* The present modification is analogous to the one introduced by Klein through the use of his "everywhere finite" differential $d\omega_z$.

§ 12. The Algebraic Functions

A single-valued function on F having no other singularities than poles yields an absolute invariant of the group G, the new function, considered in K, being analytic except for poles.

The necessary and sufficient condition which $2n$ points of $\mathfrak{F}$, σ_k and τ_k, $k = 1, \cdots, n$, must satisfy if they are to be the zeros and poles of such a function $F(t)$, is found by contour integration* to be the following:

$$\text{(i)} \qquad w_\alpha(\sigma_1)+\cdots+w_\alpha(\sigma_n)\equiv w_\alpha(\tau_1)+\cdots+w_\alpha(\tau_n) \quad \text{(mod. periods)},$$

i. e.,

$$\text{(i}'\text{)} \qquad \sum_{k=1}^{n} w_\alpha^{\sigma_k\tau_k} = \mu_\alpha \pi i + \sum_{j=1}^{p} \nu_j\, a_{\alpha j},$$

where $\alpha = 1, \cdots, p$ and the μ_α, ν_j are integers.

If one of the above points, as τ_n, be replaced by an equivalent point under the group G, the coefficients μ_α, ν_j will thereby be modified. It is evidently possible to choose the latter point so that these coefficients will all be zero, and we suppose this done.†

$$\text{(i}''\text{)} \qquad \sum_{k=1}^{n} w_\alpha^{\sigma_k\tau_k} = 0,$$

where, now, $\sigma_1, \cdots, \sigma_n$ and $\tau_1, \cdots, \tau_{n-1}$ lie in $\mathfrak{F}$, while τ_n may not.

More generally, the points $\sigma_1, \cdots, \sigma_n$ and $\tau_1, \cdots, \tau_n$ may be any $2n$ points satisfying (i$''$) and such that the points of $\mathfrak{F}$ equivalent under G to $\sigma_1, \cdots, \sigma_n$ are all distinct from the points of $\mathfrak{F}$ equivalent to $\tau_1, \cdots, \tau_n$.

To express the function $F(t)$, or to prove the existence of such a function, the procedure is similar to that in the elliptic case, where $\sigma(t-\tau)$ corresponds to the present $\Omega(t, \tau)$. Form the

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* The integrals which are to be extended round the boundary of σ are $\frac{1}{2\pi i}\int_C d \log F(t)$, $\frac{1}{2\pi i}\int_C w_\alpha(t)\, d \log F(t)$.

† It is sufficient for the application which follows that merely the ν_j be reduced to zero.

Noether's normal curve C is given by the equations:

$$\rho x_\alpha = \Phi_\alpha(t), \qquad \alpha = 1, \cdots, p.$$

The hyperelliptic case being excluded, the curve has no multiple points. The treatment by means of the functions $\Phi_\alpha(t)$ is simple and complete.

By the aid of these functions, too, a canonical Riemann's surface which Klein* has introduced can be treated satisfactorily, analytic proofs replacing the customary algebraic assumptions. Klein projects the curve C on a pencil of planes, i. e., he sets

$$z = \frac{u_\Phi}{v_\Phi},$$

where

$$u_\Phi = u_1 \Phi_1(t) + \cdots + u_p \Phi_p(t),$$

the u_k's and v_k's being non-specialized constants. As the other variable, s, he takes the following:

$$s = \frac{\dfrac{d}{dt}\left(\dfrac{u_\Phi}{v_\Phi}\right)}{v_\Phi}.$$

A further application is one that Klein† has given. The principle of correspondence in algebraic geometry, due to Charles-Cayley-Brill, was proven under suitable restrictions by Hurwitz‡ with the aid of the theta functions of several variables. Klein constructs a proof by means of his prime function along the lines of Hurwitz's proof, and the present prime function $\Omega(t, \tau)$ yields the same results. Both in Hurwitz's and in Klein's proofs details are omitted which can be satisfactorily supplied by the theorems above considered in II, § 6.

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* *Math. Ann.*, 36 (1889), p. 23. This particular canonical surface is obtained from Noether's normal curve and was given by Klein in his lectures on Abelian Functions, Göttingen, W.-S., 1888/89.

† Göttingen lectures on Abelian Functions, W.-S., 1888/89, Lecture IX, Dec. 15, 1888.

‡ *Math. Ann.*, 28 (1887), p. 561.

To uniformize the homogeneous configuration corresponding to (B) it is sufficient to set

$$z_1 = \rho\,\Omega(t, a_1) \cdots \Omega(t, a_n),$$

$$z_2 = \rho\,\Omega(t, b_1) \cdots \Omega(t, b_n),$$

$$w_1 = \sigma\,\Omega(t, \alpha_1) \cdots \Omega(t, \alpha_m),$$

$$w_2 = \sigma\,\Omega(t, \beta_1) \cdots \Omega(t, \beta_m).$$

A fundamental domain is given by the cylindrical region $(R, S, \mathfrak{F})$ corresponding to the points (ρ, σ, t), where $0 < |\rho| < \infty$, $0 < |\sigma| < \infty$, and t lies in $\mathfrak{F}$. Aside, then, from the fact that $\mathfrak{F}$, regarded as a fundamental domain for the automorphic group G, is of deficiency $p > 1$, there is the additional circumstance that both R and S are doubly connected regions.

The number of independent variables of the allied homogeneous configuration is 3.

To uniformize the configuration which corresponds to (A) let

$$\Omega(t, c_1) \cdots \Omega(t, c_l)$$

be the least common multiple of the denominators of z and w in (2). Then we set

$$x_1 = \rho\,\Omega(t, c_1) \cdots \Omega(t, c_l),$$

and x_2, x_3 are given by multiplying ρ by a suitable product of Ω-factors.

A fundamental domain is here furnished by the cylindrical region $(R, \mathfrak{F})$, corresponding to the points (ρ, t), where $0 < |\rho| < \infty$ and t lies in $\mathfrak{F}$.

In case the product that represents x_1 contains factors not appearing in the product for z_1, the number-pairs (z_1, z_2) will not stand in a one-to-one and continuous relation to the number-pairs (x_1, x_2), since the latter will contain the number-pair $(0, 0)$,—a fact of importance in the theory of homogeneous variables.

Such uniformizations as the foregoing are of use in studying

function

$$\Phi(t) = \frac{\Omega(t, \sigma_1) \cdots \Omega(t, \sigma_n)}{\Omega(t, \tau_1) \cdots \Omega(t, \tau_n)}.$$

This function has the desired zeros and poles. Moreover,

$$\Phi(t_a) = \Phi(t),$$

$$\Phi(t_{p+a}) = e^{2\sum_{k=1}^{n} w_a^{\sigma_k \tau_k}} \Phi(t) = \Phi(t),$$

where $\alpha = 1, \cdots, p$, and this completes the proof.

§ 13. Parametric Representation of a Homogeneous Algebraic Configuration

Let

$$F(w, z) = 0 \tag{1}$$

be an irreducible algebraic equation. Let homogeneous variables be introduced. If, for example, we wish to regard (1) as a curve in the projective plane, we shall set

$$z = \frac{x_2}{x_1}, \qquad w = \frac{x_3}{x_1}. \tag{A}$$

On the other hand, we may regard (1) as a curve in the plane of analysis, and set*

$$z = \frac{z_2}{z_1}, \qquad w = \frac{w_2}{w_1}. \tag{B}$$

Let (1) be uniformized as above by the function

$$z = \varphi(t).$$

Then we can express z, w in terms of t by the aid of the prime function as follows:

$$z = \frac{\Omega(t, b_1) \cdots \Omega(t, b_n)}{\Omega(t, a_1) \cdots \Omega(t, a_n)}, \qquad w = \frac{\Omega(t, \beta_1) \cdots \Omega(t, \beta_m)}{\Omega(t, \alpha_1) \cdots \Omega(t, \alpha_m)}. \tag{2}$$

Here, the points of $\mathfrak{F}$ equivalent to $a_1, \cdots, a_n$ are distinct from those of $\mathfrak{F}$ equivalent to $b_1, \cdots, b_n$; and similarly for $\alpha_1, \cdots, \alpha_m$ and $\beta_1, \cdots, \beta_m$.

* Cf. Riemann, Abelsche Funktionen, § 6; *Werke*, 1. ed., p. 103.

$$s_1 = \frac{\alpha s + \beta}{\gamma s + \delta}.$$

Thus s is homomorphic, § 5.

The present class of equations contains $3p - 3$ parameters, and it is a problem conceived in the spirit of Riemann's theory to find a vital property of the solutions which shall suffice completely to determine these parameters and thus single out from the class a unique member.

Klein* has given the following solution of this problem.

Let F be rendered simply connected by a system of cuts, the bounded surface being denoted by F'. Then the function (1) will map F' on a simply connected region of the s-plane. This region will have no branch points; but it may overlap itself, and even if this were not the case, the further regions obtained by allowing z to cross the boundary of F' and then describe F' again may conceivably overlap one another. Let the totality of such regions be denoted by Σ.

As a first restriction on the present class of differential equations Klein demands that Σ shall be simple, i. e., consist of a single-sheeted region.

To state the requirement in another form, it is this. The function s is multiple-valued on the closed surface F. And now we demand that the values which s takes on in a given point of F shall all be distinct, no matter where this point be chosen.

We arrive, then, at a class of differential equations among those under consideration whose allied function s is such that, by means of it, the algebraic configuration $f(u, z) = 0$ can be uniformized. The functions of s that here present themselves, namely u and z, are single-valued automorphic functions. But the differential equation (A) is still not uniquely determined.

The final requirement is this. The region Σ shall consist of the interior of a circle. It is still possible to pass from one circle in

* *Math. Ann.*, 19 (1882), p. 565; 20 (1882), p. 49; 21 (1883), p. 141. Cf. also Klein's Göttingen lectures, Ueber lineare Differentialgleichungen zweiter Ordnung, 1894 (lithographed).

the configurations in the space of the homogeneous variables which are allied to a given algebraic configuration.

§ 14. Linear Differential Equations on an Algebraic Configuration, and the Factor $\varphi'(t)$

In his further development of Riemann's programme relating to the determination of linear differential equations by their monodromic group Klein has studied differential equations of the type

$$(A) \qquad \frac{d^2U}{dz^2} + P(u, z)\frac{dU}{dz} + Q(u, z)U = 0,$$

where the coefficients P and Q are single-valued functions on a given algebraic Riemann's surface F corresponding to the irreducible algebraic equation $f(u, z) = 0$ of arbitrary deficiency p; the coefficients having no other singularities on F than poles, and being, therefore, rational in u, z. Let P and Q be further so restricted that the singular points of (A) are all regular, and let $p > 1$. Among such differential equations the subclass is of especial interest whose members have no singular points whatever. If U_1 and U_2 be two linearly independent solutions of such an equation and we set*

$$(1) \qquad \frac{U_2}{U_1} = s,$$

then the neighborhood of an arbitrary point of F is mapped by this function on the smooth neighborhood of a corresponding point of the extended s-plane (or sphere).

The function s is multiple-valued on F. When z describes a closed path on F, a given determination of s, continued analytically along this path, goes over into a linear function of the initial determination:

* The notation s was used by Schwarz in similar cases, to whom are due the earliest investigations in this field which appeared after Riemann's fundamental memoirs. Klein uses the letter η in this sense.

the s-plane to any other by a linear transformation of s. If, however, we regard all such differential equations (A) as equivalent,—their Schwarzian resolvent

$$[s]_z = R(u, z)$$

will in fact be the same for all,—we have the result that the differential equation (A) is uniquely determined by the above requirements.*

Thus it appears that, when an algebraic function of deficiency $p > 1$ is given, a differential equation corresponding to it can be so chosen that s is precisely the function which we obtained by conformal mapping as t, namely (13) in § 5.

From the foregoing developments we conclude that the Abelian integrals corresponding to F, when considered in their dependence on their parameters, form a class of functions which, in important respects, is incomplete. The factor $\varphi'(t) = 1/(dt/dz)$ is an essential accessory, and this constituent is supplied by the linear differential equation (A), whose parameters are determined in the spirit of Riemann and from a point of view similar to that which has dominated a long line of important researches in another branch of modern analysis,—I refer to the theorems of oscillation of Sturm and Klein.

* The demand that Σ be a circle is not the only one which leads to familiar functions. Thus we might have demanded that the boundary of Σ consist of a discrete set of points,—*discrete*, as this term is defined in the author's paper in the *Annals of Math.* (2), 14 (1913), p. 143. We should then have been led to the automorphic functions of the Schottky type. Again, the differential equation (A) would have been uniquely determined.